# Viscosity Solutions on Ramified Spaces

Dirk Schieborn

October 21, 2005

# Contents

# Chapter 1

# Introduction

In the vast field of partial differential equations, many crucial contributions of new, ingenious approaches have lead to the great variety of methods and techniques which it enjoys today. In particular the theory of nonlinear equations, which is far more scattered and less comprehensive than the linear theory, depends upon the development of new methods helping to gain deeper insights and to establish new points of view.

In general, boundary or initial value problems of nonlinear equations, among which we highlight the Hamilton-Jacobi equations as prominent examples, fail to have smooth solutions on a given domain or for all times. A method to overcome this problem is to soften the demands and to introduce appropriate concepts of *weak solutions*. The present thesis is concerned with a particularly important contribution in this spirit: the *theory of viscosity solutions*, which was initiated by S. N. Kružkov and first established in its present form by M. C. Crandall and P.-L. Lions. Similar to the idea of weakly differentiable functions known from Sobolev's theory, the concept of viscosity solutions is a generalization of classical solutions, allowing for not necessarily differentiable functions to be possible solution candidates. Roughly speaking, to be continuous is the only immediate condition a viscosity solution has to satisfy, whereas the crucial requirement – the "test function condition" – is of indirect nature: The solution has to be resistant against each smooth test function touching it by above or below, in the sense that the latter (rather than the solution itself) has to satisfy a corresponding differential inequality. Obviously, the use of test functions forms the link to the Sobolev theory of weak solutions of linear equations in divergence form. The latter, however, employs integration by parts in order to "shift" the derivatives to test functions, whereas the theory of viscosity solutions exploits the maximum principle for the same purpose. It turns out that this simple and elegant concept provides existence and comparison results for a broad class of nonlinear partial differential equations which in general do not allow for classical solutions.

The theory of viscosity solutions has been extensively studied and refined by many authors, and among the numerous contributions in the literature one can find different adaptations to more general settings. In the present thesis we refrain from a new discussion of the old subject. In contrast, we suggest a meaningful transfer of the existing theory to a setting which has not yet been treated in this context: *ramified spaces*.

Several physical phenomena such as interaction of different media can be translated into mathematical problems involving differential equations which are not defined on connected manifolds as usually, but instead on so-called ramified spaces. The latter can be roughly visualized as a collection of different manifolds of the same dimension (*branches*) with certain parts of their boundaries identified (*ramification space*). The simplest examples are *topological networks*, which basically are graphs embedded in Euclidean space. Interaction problems can be modelled by a collection of differential equations describing the behaviour of physical quantities on the branches, along with certain *transition conditions* governing the interaction of the quantities across the ramification spaces. From a mathematical point of view, transition conditions are an essential new aspect when searching for solutions. Since the year 1980, many works have been published treating different kinds of interaction problems involving linear and quasilinear differential equations. However, as far as we know, fully nonlinear equations such as Hamilton-Jacobi equations have not yet been examined to a similar extent on ramified spaces. Therefore a major goal of the present thesis is to establish a theory of viscosity solutions of first order Hamilton-Jacobi equations on ramified spaces, where the main emphasis will be placed on topological networks. In doing so, mathematical *models for granular matter* applied to ramified spaces will serve us as an motivational and illustrating example.

A closely related aim of interest in this context is the so-called *method of vanishing viscosity*, which origins in fluid dynamics and has eventually lead to the modern notion of viscosity solutions. Although the latter has been extended to second order equations – under replacement of the test functions condition by an alternative involving set-valued generalized differential operators (semi-jets) –, its original nature is of *first order type*. In this case the theory is strongly related to the origin of the concept, which gave rise to the terminology: the idea of converting a nonlinear first order equation

$$H(Du(x), u(x), x) = 0$$

into a semilinear second order equation by adding a "viscosity term" $\varepsilon\Delta u$, followed by a passage to the limit $\varepsilon \to 0$ ("vanishing viscosity"). Heuristically speaking, the viscosity term prevents the solution from immediately responding to the equation, but instead to display a smoothed behaviour. The subsequent reduction of viscosity then gradually takes away this smoothening effect and makes the solution react more quickly. The mathematical motivation is to replace the original nonlinear problem by a family of semilinear ones which can be treated with standard semilinear theory. In some sense, the difficulty of the problem is thus transferred to the question if the $\varepsilon$-family of solutions converges. In many cases this question can be positively answered by means of compactness arguments, whose essence consists in certain *a priori*-estimates. This so-called method of vanishing viscosity in general acts as a *selection principle* in the following sense: Whereas classical solutions of boundary value problems involving nonlinear equations do not exist in general, the situation is different if the demands are relaxed and weak (i.e. "almost everywhere") solutions are admitted, in which case solutions do exist but may not be unique anymore. Hence in either case the problem is unsatisfactory, as existence and uniqueness certainly are the two main features of a useful theory. Then a possible solution to this dilemma is provided by the method of vanishing viscosity, as it selects a unique weak solution. In fact it turns out that what at first might seem to be a purely formal restriction to a certain solution motivated by a technical selection procedure will end up being the "correct" solution in several other aspects, both physical and mathematical. For instance, the viscosity term appears naturally in fluid dynamics as the physical viscosity of the fluid, and the vanishing viscosity method describes limiting cases where this viscosity is neglected. On the other hand, it can be interpreted as a gradual reduction of the effect of diffusion in reaction-diffusion scenarios. On the other hand, a mathematical justification is given by the very fact, that the limit function selected by the vanishing viscosity method coincides with the viscosity solution of the original problem. Conversely, the characterization by test functions is nothing else than an appropriate intrinsic characterization of the vanishing viscosity limit. In the present thesis we will encounter the vanishing viscosity method in different contexts, starting from explicit calculations in exemplary cases and ending with general convergence results on networks.

As already mentioned, in order to transfer the test function concept to ramified spaces, we will repeatedly invoke a physical motivation: mathematical models for granular matter. In fact we consider the problem of determining the contours of maximal volume configurations of homogeneous granular material placed upon ramified domains. Let us briefly elaborate on this idea. Equilibrium configurations of dry and homogeneous granular matter can only form "heaps" the local steepness of which does not exceed a certain *angle of repose* $\alpha$ specific to the respective material. Consequently, the function describing the contours of maximal volume configurations on unramified domains without rim (such as tables) is selected among all "almost everywhere" solutions of the eikonal equation

$$H(Du(x), u(x), x) = |Du(x)| - \tan\alpha = 0,$$

with $u \equiv 0$ on the boundary. Hence another selection criterion occurs: the additional constraint that the volume functional be maximized (confer [1]). Now the connection to the theory of viscosity solutions is given by the observation that the "maximal volume solution" coincides – up to multiplication with a constant – with the distance function to the boundary. The distance function, on the other hand, can be shown to be the unique viscosity solution of the above boundary value problem and thus to be selected by the method of vanishing viscosity. In fact we demonstrate that construction methods for maximal volume solutions introduced in [1] are closely related to similar methods for viscosity solutions. Now having in mind both the granular matter and the viscosity solution interpretation of the distance function on non-ramified domains, the passage to viscosity solutions on ramified spaces suggests itself. Let us illustrate this by means of the simplest case of a ramified space: a one-dimensional topological network or a graph. In this setting, the distance function to a given collection of *boundary vertices* in fact describes the maximal volume configuration of granular matter for the following scenario: The network can be pictured as a maze in a horizontal plane, whose paths are connected at the vertices. Now think of the paths to be bounded on both sides by thin, sufficiently high glass walls perpendicular to the plane. Let us then uniformly pour as much sand as possible into the space between the glass walls, assuming that sand can run out of the maze at the boundary vertices. At the other vertices (called *transition vertices*), sand is interchanged between different connected paths. Several sand heaps will grow, each two of them separated by at least one boundary vertex. Finally the heaps stop growing and reach an equilibrium configuration. By this time each additional sand portion locally violates the angle of repose and is thus forced to leave the maze at the boundary vertices. Hence the contours of the equilibrium configuration are mathematically described by a continuous function defined on the network which vanishes at the boundary points, maximizes the volume functional, and satisfies the eikonal equation almost everywhere on the edges. Analogously to the non-ramified case, the distance function satisfies all these conditions. Hence it is selected by the maximal volume problem among all other weak solutions of the eikonal equation on the network. Now the equivalence of the two selction principles (maximal volume and viscosity solution) in the case of the eikonal equation on non-ramified domains gives reason to the question if there is a generalized test function condition which is satisfied by the distance function on networks and which might transfer the theory of viscosity solutions to ramified spaces for a certain class of Hamilton-Jacobi equations.

As a main result of the present thesis it will turn out that indeed there is a class of first order Hamilton-Jacobi equations of *eikonal type* for which the concept of viscosity solutions can be appropriately extended to networks. In fact we propose an intrinsic test function characterization for viscosity solutions on ramified spaces and justify it in different aspects: The reason why viscosity solutions are so convincing in the non-ramified case is the fact that they coincide with vanishing viscosity limits and that they entail a variety of technical advantages such as elegant comparison, uniqueness, and existence results. We show that our theory preserves all these features. In addition, we propose an adaptation of the generalized theory to certain higher dimensional ramified spaces, so-called LEP-spaces (locally elementary polygonal ramified spaces).

In view of the generalization to ramified spaces, the extension of the method of vanishing viscosity is another interesting problem, which we will treat in detail for the case of networks. It in fact requires special care at the ramification spaces, as after adding the viscosity term it turns out that the solution of the corresponding semilinear boundary value problem is only unique, if we – apart from continuity – impose a further condition on the ramification spaces: the classical Kirchhoff condition, establishing a relation between the outer normal derivatives of the solutions on the branches. The Kirchhoff condition can in fact be thought of as an extension of the “averaging effect” of the viscosity term on the branches to the ramification spaces. This averaging effect can also be physically interpreted in terms of granular matter models, where it represents local perturbations caused by small spontaneous avalanches.

Another, somewhat exceptional part of the present thesis is concerned with the investigation of certain properties of the distance function on networks. It is well known (cf. [2]) that for a convex two-dimensional domain the length of its boundary is related to the curvature functional of the distance function. In fact they are equal up to a normalizing factor independent of the choice of the domain. The distance function thus plays an important role as a link between the eikonal equation and the topology of the domain by connecting local and global concepts, a phenomenon which we will show to appear also in the context of networks. We examine the curvature functional of the distance function to boundary vertices, which is given by the number of local maxima of the distance function, where maxima have to be suitably counted at transition vertices. It turns out that this number is equal to a purely topological quantity depending only on basic graph theoretical properties of the underlying network.

## 1.1 Logical organization and chapter summary

The logical relations between the different topics presented in this thesis are manifold. However, we have decided to basically arrange the chapters according to an increasing complexity of the underlying domains, which are: one-, two-, and $n$-dimensional non-ramified domains, one-dimensional ramified domains (networks), and, finally, $n$-dimensional ramified domains (LEP spaces). For a given type of domain, we have tried to structure the material with respect to logical and/or historical consequence. We think that the possible drawback of certain topics such as the vanishing viscosity method being revisited at different stages is compensated by the coherence regarding the complexity.

In **chapter 2** we give a historical overview of the concept as well as the definitions of Kružkov's generalized solutions and (classical) viscosity solutions. We examine the vanishing viscosity method for the eikonal equation on several domains (interval, square, general domain) and show its convergence to the distance function.

In **chapter 3** we consider the eikonal equation on two-dimensional domains and connect the granular matter methods developed in [1] to existing techniques from the theory of viscosity solutions.

In **chapter 4** we give a general definition of ramified spaces and show the convergence of a generalized vanishing viscosity approach on networks. We first examine the eikonal equation and then general Hamilton-Jacobi equations of eikonal type.

In **chapter 5** we extend the theory of viscosity solutions to networks and prove uniqueness and existence results. We also show the consistency of the concept with the method of vanishing viscosity on networks.

In **chapter 6** we prove a graph theoretical result about the distance function on topological networks, connecting its curvature functional to the topology of the graph. The result is related to the concept of cycle rank of graphs and is interpreted in the language computer scientifical aspects.

In **chapter 7** we generally introduce higher dimensional ramified manifolds and define the so-called LEP spaces, to which we extend the results of chapter 5.

## 1.2 Acknowledgements

I would like to express my gratitude to Prof. Hadeler for giving me the opportunity and the support to write the present thesis, for his interest in the subject, and for allowing me great latitude in choosing the main foci. I also want to thank Helge Herr for motivational support and valuable discussions, as well as Dr. Joel Braun, Dr. Christina Kuttler, and David Pricking. Furthermore I would like to thank everyone else in the institute of biomathematics for providing an inspiring and warm working atmosphere. Finally I thank all of my friends and my parents for their patience and care.

# Chapter 2

# Viscosity solutions: history and examples

**Summary.** The purpose of the present chapter is to discuss two important concepts which contributed to the modern notion of viscosity solutions: Kružkov's generalized solutions and the method of vanishing viscosity. We investigate the latter applied to the eikonal equation in various domains and show that it converges to the distance function.

## 2.1 Introduction

In the year 1981 the notion of viscosity solutions of nonlinear first order equations, or Hamilton-Jacobi equations, appeared in the literature for the first time, when Michael G. Crandall and Pierre-Louis Lions published their papers "Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi de premier order" [3] and "Viscosity solutions of Hamilton-Jacobi equations" [4]. Although the definition given in these publications reads simple and elegant, it is nevertheless important to point out that it represents the essence of a development over a long time.

Already in 1975 S. N. Kružkov proposed a concept of generalized solutions of Hamilton-Jacobi equations of eikonal type [5], emanating from the observation that a general theory for these equations entails a dichotomic difficulty: Whereas a classical theory fails as general existence of solutions cannot be established, a weak theory cannot ensure uniqueness. Kružkov solved this problem by imposing a further, physically meaningful, constraint on the weak approach. Essentially, he demanded the existence of a uniform lower bound of the second order difference quotients of a solution candidate. And indeed, this additional requirement enabled him to overcome the problem of uniqueness and to pave the way to a general theory for a large class of Hamilton-Jacobi equations of "eikonal type". As a matter of fact, Kružkov's theory is not only justified by existence and uniqueness results, but instead his notion of a generalized solution has a physical interpretation related to the classical principles of Fermat and Huygens in geometric optics (cf. [5]), which, in turn, are strongly connected to the eikonal equation. This reflects the fact that his generalized solutions only make sense for a Hamilton-Jacobi equations which are related to the eikonal equation.

As mentioned in [6], "analogies of S. N. Kružkov's theory of scalar conservation laws ([7]) provided guidance for the notion [of viscosity solutions] and its presentation". In fact, inspired by some of the essential ideas in Kružkov's work, Crandall and Lions found a strikingly simple intrinsic representation of his generalized solutions, leading to the notion of viscosity solutions of first order equations [4]. Its publication initiated avid research activities, triggering the discovery of a chain of related and much more general results. Later on, Lions discovered a possibility to extend the concept to second order equations, modifying the definition such that they only bear little resemblance to their original version. His achievement was to prove a maximum principle and a corresponding uniqueness result for viscosity solutions of convex nonlinear second order Hamilton-Jacobi equations by means of stochastic control theory, a result which was extended to fully nonlinear second order elliptic equations by R. Jensen [8] five years later.

However, as abstract the recent general definitions of viscosity solutions might be, one basic feature always plays a fundamental role: the possibility of approximating the solution by the method of vanishing viscosity. We begin with a description of this method, followed by three concrete examples for its application to the eikonal equation. After that we elaborate on Kružkov's solutions and end with the modern definition of viscosity solution according to Crandall and Lions.

## 2.2 The idea of vanishing viscosity

As mentioned above, the method of vanishing viscosity is a procedure of selecting a certain weak solution of a first order nonlinear problem which in general has no classical solution. The idea is to slightly modify the original problem to get a semilinear problem, whereby the extent of the modification is controlled by a parameter $\varepsilon$. Now the existence and uniqueness results available in the semilinear theory yield a unique, sufficiently regular solution $u_\varepsilon$ for each value $\varepsilon > 0$. Then the modification is "undone" by taking the limit $\varepsilon \to 0$, leaving the question if and in what sense the family of functions $u_\varepsilon$ converges to a limit function called the "vanishing viscosity solution".

Let us put the idea in concrete mathematical terms: Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and consider a first order Hamilton-Jacobi equation of the form

$$\begin{cases} H(Du(x), u(x), x) = 0 & \text{on } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

where $H : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a nonlinear function, the so-called *Hamiltonian*. The "viscous" modification of the problem, depending on the parameter $\varepsilon$, is given by

$$\begin{cases} \varepsilon\Delta u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon) = 0 & \text{on } \Omega \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

and one has to prove the convergence of the functions $u_\varepsilon$ to a limit function $u$ as $\varepsilon \to 0$, which can be achieved by establishing suitable *a priori* estimates.

## 2.3 Vanishing viscosity applied to the eikonal equation

At this stage we postpone a general approach, and instead illustrate the method of vanishing viscosity by means of several exemplary boundary value problems of the eikonal equation. Accordingly, on a bounded domain $\Omega$ we consider the boundary value problem

$$\begin{cases} |Du| - 1 = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Clearly a classical solution does not exist. However, there are infinitely many Lipschitz continuous functions satisfying the boundary condition and whose modulus of gradient – existing almost everywhere by the theorem of Rademacher – equals 1, possibly except for a set of measure zero. Obviously the distance function $d$ to the boundary is contained in this class of *weak* or *"almost everywhere"* solutions, and we will demonstrate by means of explicit calculations how it is selected by the method of vanishing viscosity.

Both for technical reasons and in order to stay compatible with the theory in subsequent chapters, we do not apply the method of vanishing viscosity to the eikonal equation itself, but to the equivalent equation

$$|Du|^2 - 1 = 0.$$

Then for $\varepsilon > 0$ the corresponding semilinear "viscous" problem reads

$$\begin{matrix} \varepsilon\Delta u_\varepsilon + |Du_\varepsilon|^2 - 1 = 0 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{matrix} \tag{2.4}$$

and the semilinear theory can be applied. In fact, for domains with sufficiently smooth boundaries we have the following

**Proposition 2.1** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and with boundary $\partial\Omega$ of class $C^{2,\alpha}$. Then the boundary value problem 2.4 has a unique solution $u_\varepsilon \in C^{2,\alpha}(\bar{\Omega})$.*

**Proof.** This is a direct consequence of theorems 9.1 and 14.8 in [9]. $\square$

We now consider three cases: an interval, a square, and an arbitrary convex domain in $\mathbb{R}^n$ with $C^2$-boundary. In either case, we take advantage of a transformation of (2.4) into a linear problem.

### 2.3.1 Convergence on the interval

**Lemma 2.1** *Let $\Omega := ]0,1[ \subset \mathbb{R}$ and let $d : \Omega \to \mathbb{R}$ be the distance function to the boundary on $\Omega$. Then for each $\varepsilon > 0$ there is a unique $C^2$-solution $u_\varepsilon$ of the boundary value problem*

$$\begin{cases} \varepsilon u_\varepsilon''(x) - (u_\varepsilon'(x))^2 + 1 = 0 \quad \text{on } \Omega \\ u_\varepsilon(0) = u_\varepsilon(1) = 0\,. \end{cases} \tag{2.5}$$

*Furthermore the functions $u_\varepsilon$ converge pointwise to $d$ on $[0,1]$ as $\varepsilon \to 0$.*

**Proof.** First observe that we can convert boundary value problem (2.5) into a linear problem by means of the following transformation. Fix $\varepsilon > 0$ and suppose that $u_\varepsilon$ solves (2.5). Then for $a := 1/\varepsilon$ the function

$$w_a := \exp(-\varepsilon^{-1} u_\varepsilon) - 1$$

satisfies the linear boundary value problem

$$\begin{cases} w_a''(x) - a^2 w_a(x) = a^2 \quad \text{on } \Omega, \\ w_a(0) = w_a(1) = 0\,. \end{cases} \tag{2.6}$$

In turn, each solution $w_a$ of (2.6) with $w_a > -1$ can be transformed into a solution of (2.5) by means of the inverse transformation

$$u_\varepsilon(x) = -\frac{1}{a}\log(w_a(x) + 1).$$

Hence the existence and uniqueness of a $C^2$-solution $u_\varepsilon$ of (2.5) follows from the fact that the linearity of (2.6) immediately yields a unique solution $w_a$, which satisfies $w_a > -1$ by the maximum principle.

Observe that the solution $w_a$ can be represented by the formula

$$w_a(x) = a^2 \int_0^1 g(x,t)\,dt,$$

where $g(x,t)$ is the Green's function of the homogeneous equation

$$u''(x) - a^2 u(x) = 0$$

on $\Omega$ vanishing at the boundary.

In other words, for each $t \in ]0,1[$ the function $g(x,t)$ satisfies the following equation in the distribution sense

$$\begin{cases} \frac{d}{dx^2} g(x,t) - a^2 g(x,t) = \delta(x-t) \quad \text{on } ]0,1[\,, \\ g(0,t) = g(1,t) = 0\,, \end{cases}$$

where $\delta$ denotes the Dirac delta function. It can be readily verified that $g(x,t)$ is given by

$$g(x,t)=\begin{cases} -\sinh ax \sinh k(1-t)(a\sinh a)^{-1}, & 0\le x<t \\ -\sinh a(1-x)\sinh at(a\sinh a)^{-1}, & t<x\le 1. \end{cases}$$

Plugging in, we get the integral equation

$$\begin{aligned} w_a(x) &= -a^2\int_x^1 \frac{\sinh ax \sinh a(1-t)}{a\sinh a}dt - a^2\int_0^x \frac{\sinh a(1-x)\sinh at}{a\sinh a}dt \\ &= -1+\frac{\sinh a(1-x)+\sinh ax}{\sinh a}. \end{aligned}$$

Observe that $w_a$ indeed solves the inhomogeneous linear problem (2.6). Resubstituting, we obtain the explicit formula for the solution of (2.5), which reads

$$u_\varepsilon(x) = -\frac{1}{a}\log(w_a(x)+1) = -\frac{1}{a}\log\left(\frac{\sinh a(1-x)+\sinh ax}{\sinh a}\right).$$

As we want to examine the behaviour of $u_\varepsilon$ for $\varepsilon\to 0$, we have to consider the limit $a=\varepsilon^{-1}\to\infty$. For this purpose we express the hyperbolic functions in terms of exponential functions and obtain

$$u_\varepsilon(x) = -\frac{1}{a}\log\left(\frac{e^{a(1-x)}-e^{a(x-1)}+e^{ax}-e^{-ax}}{e^a-e^{-a}}\right) = \frac{1}{a}\left(ax-\log\left(\underbrace{\frac{(e^a-1)(e^{a(2x-1)}+1)}{e^a-e^{-a}}}_{(1)}\right)\right).$$

By means of this expression it is easy to see that for $0\le x<1/2$ the term (1) tends to 1 as $a\to\infty$. On the other hand, by choosing a different representation we get

$$u_a(x) = -\frac{1}{a}\left(ax-\log\left(\underbrace{\frac{e^a-e^{-a}}{e^{a(1-2x)}-e^{-a}+1-e^{-2a}}}_{(2)}\right)\right).$$

Now, for $1/2<x\le 1$, expression (2) obviously behaves like $e^a$ as $a\to\infty$. Altogether it follows

$$\lim_{a\to\infty} u_a(x) = \begin{cases} x & 0\le x<1/2 \\ 1-x & 1/2<x\le 1, \end{cases}$$

which means that the $u_a$ pointwise converge to the distance function $d$ on $\Omega$, as asserted. □

**Lemma 2.2** *We have*

$$\lim_{\varepsilon\to 0} \frac{d}{dx} u_\varepsilon(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ 0 & \text{if } x = 1/2 \\ -1 & \text{if } 1/2 < x \le 1. \end{cases}$$

**Proof.** Again we set $a := 1/\varepsilon$. A direct calculation yields

$$\begin{aligned} \frac{d}{dx} u_a(x) &= -\frac{-\cosh a(1-x) + \cosh ax}{\sinh a(1-x) + \sinh ax} \\ &= -\frac{-e^{a(1-x)} - e^{a(x-1)} + e^{ax} + e^{-ax}}{e^{a(1-x)} - e^{a(x-1)} + e^{ax} - e^{-ax}} = \frac{e^{a(1-2x)} + e^{-a} - 1 - e^{-2ax}}{e^{a(1-2x)} - e^{-a} + 1 - e^{-2ax}}. \end{aligned}$$

The assertion is immediate when letting $a \to \infty$ in the last term. □

A combination of lemma 2.1 and lemma 2.2 yields the following

**Corollary 2.1** *The functions $u_\varepsilon$ converge to $d$ with respect to the $C^1$-norm on $\Omega$.*

### 2.3.2 Convergence on a square

We now apply the method of vanishing viscosity to the eikonal equation on a square and prove an analogous statement.

**Lemma 2.3** *For each $\varepsilon > 0$ there is a unique solution $u_\varepsilon$ of the boundary value problem*

$$\begin{cases} \varepsilon\Delta u_\varepsilon - |Du_\varepsilon|^2 + 1 = 0 & \text{on } \Omega := ]0,1[\times]0,1[ \\ u_\varepsilon \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

*Furthermore the functions $u_\varepsilon$ converge pointwise to the distance function $d$ on $\Omega$ as $\varepsilon \to 0$.*

For the proof of lemma 2.3 we need the following well-known facts about the spectral representation of Green's functions for the Dirichlet function on arbitrary domains, associated with a self-adjoint linear operator.

**Lemma 2.4** *The Green's function $G(p,q)$ for the Dirichlet problem associated with the self-adjoint linear operator $L$ on a bounded domain $\Omega \subset \mathbb{R}^m$, $m \in \mathbb{N}$, has the form*

$$G(p,q) = \sum_{n=1}^{\infty} \frac{u_n(p)u_n(q)}{\lambda_n},$$

*where $u_n$, $n \ge 1$, are orthonormal eigenfunctions of $L$ corresponding to the eigenvalues $\lambda_n$ defined by*

$$\begin{aligned} &Lu_n(p) = \lambda_n u_n(p), && p \in \Omega \\ &u_n(p) = 0, && p \in \partial D. \end{aligned}$$

**Lemma 2.5** *Let $\kappa \in \mathbb{R}$. If $L$ is replaced by $L - \kappa$ in lemma 2.4, the Green's functions takes the form*

$$G(p,q) = \sum_{n=1}^{\infty} \frac{u_n(p)u_n(q)}{\lambda_n - \kappa},$$

*provided $\kappa$ is not an eigenvalue of $L$.*

**Proof.** (of lemma 2.4 and lemma 2.5) Confer theorem 9.4 and its corollary in [10]. □

**Proof.** (of lemma 2.3) We apply the same transformation as in the proof of lemma 2.1. Accordingly, if $u_\varepsilon$ solves (2.7), then the function

$$w_a := \exp(-\varepsilon^{-1}u_\varepsilon) - 1, \quad a := 1/\varepsilon,$$

solves the linear boundary value problem

$$\begin{cases} \Delta w_a - a^2 w_a = a^2 & \text{on } \Omega \\ w_a \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Conversely, it is well-known that for each $a > 0$ there exists a unique solution $w_a$ of (5.6). Analogously to the proof of 2.1 we verify $w_a > -1$ and apply the inverse transformation to find that a solution $u_\varepsilon$ of (5.1) exists uniquely.

Observe now that $w_a$ may be represented by the formula

$$w_a(p) = a^2 \int_\Omega G(p,q)\,dq,$$

where the integral kernel $G$ is the Green's function associated with the homogeneous equation

$$\Delta u - a^2 u = 0$$

on the square vanishing on the boundary. Accordingly, $G$ formally satisfies the equation

$$\Delta G(p,q) - a^2 G(p,q) = \delta(p-q), \quad p,q \in \Omega.$$

Now observe that the functions

$$u_{m,n}(x,y) := \frac{1}{16}\sin(m\pi x)\sin(n\pi y)$$

form a complete orthonormal system of eigenfunctions of $-\Delta$ vanishing on the boundary of $\Omega$, corresponding to the eigenvalues

$$\lambda_{m,n} := \pi^2(m^2 + n^2).$$

Accordingly, setting $L := -\Delta$ and $\kappa := -a^2$, we obtain by virtue of lemma 2.5

$$G(p,q) = G(x_1,y_1,x_2,y_2) = \sum_{m,n=1}^{\infty} \frac{\sin(m\pi x_1)\sin(n\pi y_1)\sin(m\pi x_2)\sin(n\pi y_2)}{\pi^2(m^2+n^2)+a^2}. \tag{2.9}$$

We now represent the solution $w_a$ of the boundary value problem 2.8 by means of the standard integral representation formula (cf. for instance theorem 9.6 in [10]) and obtain

$$\begin{aligned} w_a(x_1,y_1) &= -\frac{a^2}{16}\sum_{m,n=1}^{\infty}\int_0^1\int_0^1 \frac{\sin(m\pi x_1)\sin(n\pi y_1)\sin(m\pi x_2)\sin(n\pi y_2)}{\pi^2(m^2+n^2)+a^2}\,dx_2\,dy_2 \\ &= -\frac{a^2}{16}\sum_{m,n=1,3,\ldots}\frac{4}{mn\pi^2}\frac{\sin(m\pi x_1)\sin(n\pi y_1)}{\pi^2(m^2+n^2)+a^2}. \end{aligned}$$

This representation is not yet suitable to gain any information about the behaviour as $a \to \infty$. For this purpose, we choose an alternative way to represent the Green's function, which will turn out to be more fruitable. In fact, according to [10], p. 270, problem 11, the above Green's function (2.9) may be expressed in the form

$$G(p,q) := 2\sum_{n=1}^{\infty}\frac{\sinh\sigma y_2\ \sinh\sigma(y_1-1)}{\sigma\ \sinh\sigma}\sin n\pi x_1\ \sin n\pi x_2$$

for the case $0 < y_2 < y_1 < 1$, with $\sigma^2 = a^2 + n^2\pi^2$, $p = (x_1,y_1)$ and $q = (x_2,y_2)$.

Applying the integral representation formula of theorem 9.6 in [10] once more, the solution $w_a$ takes the form

$$\begin{aligned} w_a(x_1,y_1) &= 2a^2\sum_{n=1}^{\infty}\int_0^1\int_0^{y_1}\frac{\sinh\sigma y_2\sinh\sigma(y_1-1)}{\sigma\sinh\sigma}\sin n\pi x_1\sin n\pi x_2 dy_2 dx_2 \\ &+ 2a^2\sum_{n=1}^{\infty}\int_0^1\int_{y_1}^1\frac{\sinh\sigma(y_2-1)\sinh\sigma y_1}{\sigma\sinh\sigma}\sin n\pi x_2\sin n\pi x_1 dy_2 dx_2. \end{aligned} \tag{2.10}$$

Evaluating the respective integrals yields

$$\begin{aligned} &\int_0^1\int_0^{y_1}\frac{\sinh\sigma y_2\sinh\sigma(y_1-1)}{\sigma\sinh\sigma}\sin n\pi x_1\sin n\pi x_2 dy_2 dx_2 \\ &= \frac{\sinh\sigma(y_1-1)\ \sin n\pi x_1}{\sigma\sinh\sigma}\int_0^1 \sin n\pi x_2\,dx_2\int_0^{y_1}\sinh\sigma y_2\,dy_2 \\ &= \frac{\sinh\sigma(y_1-1)\ \sin n\pi x_1}{\sigma\sinh\sigma}\cdot\frac{((-1)^{n+1}+1)}{n\pi}\cdot\frac{1}{\sigma}(\cosh\sigma y_1 - 1) \\ &= \begin{cases} \frac{2\sinh\sigma(y_1-1)\ \sin n\pi x_1}{n\pi\sigma^2\sinh\sigma}\cdot(\cosh\sigma y_1 - 1) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{2.11}$$

and

$$
\begin{aligned}
&\int_0^1 \int_{y_1}^1 \frac{\sinh \sigma(y_2-1)\sinh \sigma y_1}{\sigma \sinh \sigma} \sin n\pi x_2 \sin n\pi x_1 dy_2 dx_2 \\
=\ & \frac{\sinh \sigma y_1 \ \sin n\pi x_1}{\sigma \sinh \sigma} \cdot \frac{((-1)^{n+1}+1)}{n\pi} \cdot \frac{1}{\sigma}(1-\cosh \sigma(y_1-1)) \\
=\ & \begin{cases} \frac{2\sinh \sigma y_1 \ \sin n\pi x_1}{n\pi\sigma^2 \sinh \sigma} \cdot (1-\cosh\sigma(y_1-1)) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
\tag{2.12}
$$

If we insert (2.11) and (2.12) into equation (2.10) and simplify, we obtain

$$
\begin{aligned}
w_a(x,y) &= 4a^2 \sum_{n=1,3,\ldots} \frac{\sin n\pi x}{n\pi\sigma^2 \sinh \sigma}[\sinh\sigma(y-1)(\cosh\sigma y-1)-\sinh\sigma y(\cosh\sigma(y-1)-1)] \\
&= -4a^2 \sum_{n=1,3,\ldots} \frac{\sin n\pi x}{n\pi\sigma^2 \sinh \sigma}[\sinh\sigma(1-y)(\cosh\sigma y-1)+\sinh\sigma y(\cosh\sigma(1-y)-1)] \\
&= -4a^2 \sum_{n=1,3,\ldots} \frac{\sin n\pi x}{n\pi\sigma^2} \left(1-\frac{\sinh\sigma(1-y)+\sinh\sigma y}{\sinh\sigma}\right).
\end{aligned}
$$

This is an expression for the solution of the transformed linear problem 2.8 to which we now apply the inverse transformation

$$
u_\varepsilon(x,y) = -\frac{1}{a}\log(w_a+1),
$$

obtaining

$$
u_\varepsilon(x,y) = -\frac{1}{a}\log\left[1-\sum_{n=1,3,\ldots}\frac{4a^2 \sin n\pi x}{\pi\sigma^2 n}\left(1-\frac{\sinh\sigma(1-y)+\sinh\sigma y}{\sinh\sigma}\right)\right].
$$

Taking advantage of the well-known relation

$$
\sum_{n=1,3,\ldots} \frac{\sin n\pi x}{n} = \frac{\pi}{4} \tag{2.13}
$$

for all $0<x<1$, we obtain using $\sigma^2 = a^2+n^2\pi^2$

$$
u_\varepsilon(x,y) = -\frac{1}{a}\log\left[\sum_{n=1,3,\ldots}\frac{4a^2 \sin n\pi x}{\pi\sigma^2 n}\left(\frac{\pi^2 n^2}{a^2}+\frac{\sinh\sigma(1-y)+\sinh\sigma y}{\sinh\sigma}\right)\right].
$$

Let us now fix $0 \le y < 1/2$ and further compute

$$\begin{aligned} u_\varepsilon(x,y) = -\frac{1}{a}\log\Bigg[e^{-ay}\Bigg(\sum_{n=1,3,\dots}\frac{4e^{ay}\pi^2n^2\sin n\pi x}{\pi\sigma^2 n} \\ + \sum_{n=1,3,\dots}\frac{4e^{ay}a^2\sin n\pi x}{\pi\sigma^2 n}\cdot\frac{\sinh\sigma(1-y)+\sinh\sigma y}{\sinh\sigma}\Bigg)\Bigg] \qquad (2.14) \\ = y - \frac{1}{a}\log\Bigg[\underbrace{\sum_{n=1,3,\dots}\frac{4e^{ay}\pi^2n^2\sin n\pi x}{\pi\sigma^2 n}}_{(1)} + \underbrace{\sum_{n=1,3,\dots}\frac{4e^{ay}a^2\sin n\pi x}{\pi\sigma^2 n}\cdot\frac{\sinh\sigma(1-y)+\sinh\sigma y}{\sinh\sigma}}_{(2)}\Bigg]. \end{aligned}$$

We analyze the behaviour of $u_\varepsilon(x,y)$ for fixed $x$ and $y$ upon letting $a = \varepsilon^{-1} \to \infty$. To this end we restrict ourselves to the case $0 \le y < 1/2$ and consider the expressions (1) and (2) separately, starting with expression (2).

Observe that we have

$$\sigma \simeq a \quad \text{for large } a \text{ and fixed } n, \qquad (2.15)$$

due to the relation $\sigma^2 = \pi^2n^2 + a^2$. By (2.15) and $0 \le y < 1/2$ we compute

$$\lim_{a\to\infty}\frac{e^{\sigma+y(a-\sigma)} - e^{-\sigma+y(a+\sigma)} + e^{(a+\sigma)y} - e^{(a-\sigma)y}}{e^\sigma - e^{-\sigma}} = \lim_{a\to\infty}\frac{e^a - e^{-a+2ay} + e^{2ay} - 1}{e^a} = 1. \qquad (2.16)$$

Now we compute the limit of (2), using the fact that limit and summation can be interchanged in this case.

$$\begin{aligned} &\lim_{a\to\infty}\sum_{n=1,3,\dots}\frac{4e^{ay}a^2\sin n\pi x}{\pi\sigma^2 n}\cdot\frac{\sinh\sigma(1-y)+\sinh\sigma y}{\sinh\sigma} \\ =\ &\sum_{n=1,3,\dots}\lim_{a\to\infty}\frac{4e^{ay}a^2\sin n\pi x}{\pi\sigma^2 n}\cdot\frac{\sinh\sigma(1-y)+\sinh\sigma y}{\sinh\sigma} \\ =\ &\sum_{n=1,3,\dots}\lim_{a\to\infty}\frac{4a^2\sin n\pi x}{\pi\sigma^2 n}\cdot e^{ay}\cdot\frac{e^{\sigma(1-y)} - e^{-\sigma(1-y)} + e^{\sigma y} - e^{-\sigma y}}{e^\sigma - e^{-\sigma}} \\ =\ &\sum_{n=1,3,\dots}\lim_{a\to\infty}\frac{4a^2\sin n\pi x}{\pi\sigma^2 n}\cdot\frac{e^{\sigma+y(a-\sigma))} - e^{-\sigma+y(a+\sigma)} + e^{(a+\sigma)y} - e^{(a-\sigma)y}}{e^\sigma - e^{-\sigma}} \\ =\ &\sum_{n=1,3,\dots}\frac{4\sin n\pi x}{\pi n}\cdot\lim_{a\to\infty}\frac{e^{\sigma+y(a-\sigma))} - e^{-\sigma+y(a+\sigma)} + e^{(a+\sigma)y} - e^{(a-\sigma)y}}{e^\sigma - e^{-\sigma}} = 1. \qquad (2.17) \end{aligned}$$

Observe that we have invoked (2.13) and (2.16) for the last equality.

Let us now consider the first expression (1). We first write

$$\sum_{n=1,3,\ldots} \frac{4e^{ay}\pi^2 n^2 \sin n\pi x}{\pi\sigma^2 n} = e^{ay} \sum_{n=1,3,\ldots} \frac{4\pi n \sin n\pi x}{\pi^2 n^2 + a^2} \tag{2.18}$$

and show that the sum

$$S_a(x) := \sum_{n=1,3,\ldots} \frac{4\pi n \sin n\pi x}{\pi^2 n^2 + a^2} \tag{2.19}$$

is the Fourier expansion of an appropriate 2-periodic function. In fact we consider the 2-periodic functions $f^{(a)} : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$, which are defined by

$$f^{(a)}(x) := \begin{cases} -e^{-a(x+1)} - e^{-a(1-(x+1))}, & \text{if } -1 \leq x \leq 0, \\ e^{-ax} + e^{-a(1-x)}, & \text{if } 0 \leq x \leq 1, \end{cases}$$

on the interval $[-1, 1]$. Observe that each $f^{(a)}$ is an odd function which is symmetric with respect to the lines

$$x = m + 1/2, \quad m \in \mathbb{Z}.$$

Hence its Fourier expansion has the form

$$f^{(a)}(x) = \sum_{n=1}^{\infty} b_n^{(a)} \sin n\pi x, \tag{2.20}$$

with $b_n^{(a)} = 0$ if $n \in 2\mathbb{Z}$. In fact when computing the $b_n^{(a)}$ according to the formula

$$b_n^{(a)} = \int_{-1}^{1} f^{(a)}(t) \sin n\pi t \, dt,$$

we obtain

$$b_n^{(a)} = \begin{cases} \frac{4\pi n(e^{-a}-1)}{\pi^2 n^2 + a^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{2.21}$$

By (2.19), (2.20), and (2.21) it follows

$$(e^{-a} - 1) S_a(x) = f^{(a)}(x).$$

This combined with (2.18) yields

$$\sum_{n=1,3,\ldots} \frac{4e^{ay}\pi^2 n^2 \sin n\pi x}{\pi\sigma^2 n} = e^{ay} \cdot \frac{e^{-ax} + e^{-a(1-x)}}{e^{-a} - 1}, \quad 0 \leq x \leq 1. \tag{2.22}$$

For large $a > 0$ we therefore obtain by virtue of (2.14), (2.17), and (2.22)

$$u_\varepsilon(x) \simeq y - \frac{1}{a}\log\left(e^{ay}\cdot\frac{e^{-ax}+e^{-a(1-x)}}{e^{-a}-1}+1\right) = y - \frac{1}{a}\log\left(\frac{e^{-a(x-y)}+e^{-a(1-x-y)}}{e^{-a}-1}+1\right).$$

Using $0 \le y < 1/2$ it is then easy to verify that we have

$$\lim_{\varepsilon\to 0} u_\varepsilon(x,y) = \begin{cases} x & \text{for } 0 \le x < y \\ y & \text{for } y < x < 1-y \\ 1-x & \text{for } 1-y < x < 1. \end{cases}$$

By reasons of symmetry it is clear that we obtain the same result for $1/2 < y \le 1$. We therefore have shown that the functions $u_\varepsilon$ converge to the distance function $d$ on the square as $\varepsilon \to 0$. $\square$

### 2.3.3 $L^1$-Convergence on convex domains with boundary of class $C^2$

The method of explicitly constructing the solutions $u_\varepsilon$ of the "viscous" eikonal equation by means of a known Green's function certainly fails for more general domains. However, we prove an $L^1$-convergence result for the case that the domain is convex and has a $C^2$-boundary.

**Lemma 2.6** *Let $\Omega \in \mathbb{R}^n$ be a domain with $C^2$-boundary. For any $\varepsilon > 0$ let $u_\varepsilon \in C^2(\bar{\Omega})$ be the unique solution of the boundary value problem*

$$\begin{cases} \varepsilon\Delta u_\varepsilon - |Du_\varepsilon|^2 + 1 = 0 & \text{on } \Omega \\ u_\varepsilon \equiv 0 & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

*which exists according to proposition 2.1. Then we have*

$$\lim_{\varepsilon\to 0}\int_\Omega |Du_\varepsilon|^2 - 1\,dx = 0.$$

The proof of lemma 2.6 is based upon the following proposition.

**Proposition 2.2** *Let $\Omega \subset \mathbb{R}^n$ be a domain with $C^2$-boundary. Then for each $a > 0$ and each $C^2$-solution $w_a$ of the boundary value problem*

$$\begin{cases} \Delta w_a - a^2 w_a = a^2 & \text{on } \Omega \\ w_a \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.24}$$

*we have*

$$\lim_{a\to\infty}\int_\Omega (w_a + 1)\,dx = 0.$$

**Proof.** (of proposition 2.2). Fix $a > 0$ and let $w_a \in C^2(\bar{\Omega})$ solve (2.24). We define

$$\Omega_a := \{y = ax \in \mathbb{R}^n \,|\, x \in \Omega\},$$

where $\Omega_1 = \Omega$. Applying the transformation $y = ax$ we observe that

$$\tilde{w}_a(y) = \tilde{w}_a(ax) := w_a(x)$$

solves the boundary value problem

$$\begin{cases} \Delta \tilde{w}_a = \tilde{w}_a + 1 & \text{on } \Omega_a \\ \tilde{w}_a \equiv 0 & \text{on } \partial\Omega_a. \end{cases} \tag{2.25}$$

Furthermore, the function $v_a := \tilde{w}_a + 1$ is a solution of

$$\begin{cases} \Delta v_a = v_a & \text{on } \Omega_a \\ v_a \equiv 1 & \text{on } \partial\Omega_a. \end{cases} \tag{2.26}$$

Observe that for all $a > 0$ we have $v_a \leq 1$ on $\bar{\Omega}_a$ by the maximum principle.

Now define for $\beta > 0$

$$\Omega^\beta := \{y \in \Omega \,|\, \operatorname{dist}(y, \partial\Omega) \geq \beta\}$$

and analogously for $a > 0$, $\beta > 0$

$$\Omega_a^\beta := \{y = ax \in \mathbb{R}^n \,|\, x \in \Omega^\beta\}.$$

Fix $\beta > 0$. We show that there is a positive function $S : \mathbb{R} \to \mathbb{R}$ with $S(a) \to 0$ as $a \to \infty$, such that

$$v_a(x) < S(a) \quad \text{for all } x \in \Omega_a^\beta. \tag{2.27}$$

To this end we define for any $r > 0$ the function $v^r$ to be the solution of the boundary value problem (2.26), where the domain $\Omega_a$ is replaced by the open ball $B_r(0)$ with radius $r > 0$. As is well-known, $v^r$ is given by a suitable Bessel function depending on the radius $r$. Hence one easily verifies that

$$v^r(0) \to 0 \quad \text{as } r \to \infty. \tag{2.28}$$

Fix now $x \in \Omega_a^\beta$. Then $B_{a\beta}(x) \subset \Omega_a$. Define the function

$$\varphi^x : y \mapsto v^{a\beta}(y - x).$$

Then we have

$$1 = \varphi^x(y) \geq v_a(y) \quad \text{for all } y \in \partial B_{a\beta}(x),$$

which implies

$$v^{a\beta}(0) = \varphi^x(x) \geq v_a(x)$$

by the maximum principle. As $x \in \Omega_a^\beta$ has been chosen arbitrarily, it follows

$$S(a) \geq v_a(x) \quad \text{for all } x \in \Omega_a^\beta,$$

where we have set $S(a) := v^{a\beta}(0)$. Furthermore, by means of relation (2.28) we have

$$S(a) \to 0 \quad \text{for } a \to \infty, \tag{2.29}$$

and (2.27) is proved.

Now fix $\varepsilon > 0$. As the boundary $\partial\Omega$ is of class $C^2$, its curvature is bounded, whence there is a constant $C > 0$ independent of $\beta$ such that

$$|\Omega \backslash \Omega^\beta| < C \cdot \beta.$$

We now choose $\beta > 0$ such that $C\beta \leq \varepsilon/2$ and compute

$$\begin{aligned} & \int_\Omega (w_a(x) + 1)\, dx = \frac{1}{a^n} \int_{\Omega_a} (\tilde{w}_a(y) + 1)\, dy = \frac{1}{a^n} \int_{\Omega_a} v_a(y)\, dy \\ \leq \;& \frac{1}{a^n} |\Omega_a \backslash \Omega_a^\beta| + \frac{1}{a^n} \int_{\Omega_a^\beta} v_a(y)\, dy \leq |\Omega \backslash \Omega^\beta| + \frac{1}{a^n} |\Omega_a^\beta| \cdot S(a) \leq \varepsilon/2 + |\Omega^\beta| \cdot S(a). \end{aligned}$$

Hence by (2.29) we have

$$\int_\Omega (w_a + 1)\, dx < \varepsilon$$

if $a$ is large enough. The assertion follows. $\square$

**Proof.** (of lemma 2.6). We apply the same transformation as in the proof of lemma 2.1. Let $u_\varepsilon \in C^2(\bar{\Omega})$ be a solution of (2.23) and set $a := 1/\varepsilon$. Then the function

$$w_a := \exp(-\varepsilon^{-1} u_\varepsilon) - 1$$

satisfies the linear boundary value problem

$$\begin{cases} \Delta w_a - a^2 w_a = a^2 & \text{on } \Omega \\ w_a \equiv 0 & \text{on } \partial\Omega. \end{cases} \tag{2.30}$$

From the inverse transformation

$$u_\varepsilon := -\frac{1}{a} \log(w_a + 1)$$

it follows

$$|Du_\varepsilon|^2 = \frac{1}{a^2} \sum_{i=1}^n \frac{(D_i w_a)^2}{(w_a + 1)^2}.$$

Using

$$D_i\left(\frac{w_a}{w_a+1}\right)=\frac{D_iw_a}{(w_a+1)^2},$$

we get by partial integration

$$\int_\Omega |Du_\varepsilon|^2\,dx=\frac{1}{a^2}\int_\Omega\sum_{i=1}^n\frac{(D_iw_a)^2}{(w_a+1)^2}\,dx=\frac{1}{a^2}\int_\Omega\sum_{i=1}^n\frac{D_iw_a}{(w_a+1)^2}D_iw_a\,dx$$

$$=\frac{1}{a^2}\int_\Omega\sum_{i=1}^n D_i\left(\frac{w_a}{w_a+1}\right)D_iw_a\,dx=-\frac{1}{a^2}\int_\Omega\sum_{i=1}^n\frac{w_a}{w_a+1}D_{ii}w_a\,dx=-\frac{1}{a^2}\int_\Omega\frac{w_a}{w_a+1}\Delta w_a\,dx,$$

where the boundary terms vanish due to $w_a\equiv 0$ on $\partial\Omega$.

On the other hand, (2.30) implies

$$a^2(w_a+1)=\Delta w_a,$$

whence it follows

$$\int_\Omega |Du_\varepsilon|^2\,dx=-\int_\Omega w_a\,dx.$$

We obtain

$$\int_\Omega(|Du_\varepsilon|^2-1)\,dx=-\int_\Omega(w_a+1)\,dx,$$

and the assertion follows by proposition 2.2. □

**Theorem 2.1** *Suppose that the domain in lemma 2.6 be convex. Then* $|Du_\varepsilon|^2\to 1$ *with respect to the* $L^1$*-norm as* $\varepsilon\to 0$.

**Proof.** Let $d:\bar\Omega\to\mathbb{R}$ be the distance function to the boundary $\partial\Omega$. We show that $u_\varepsilon\le d$ on $\bar\Omega$ for all $\varepsilon>0$. For this purpose assume this were not the case. Then there is an $\varepsilon>0$ and a point $x_0\in\Omega$ such that $u_\varepsilon(x_0)>d(x_0)$. Let $y\in\partial\Omega$ satisfy $|x_0-y|=\min_{z\in\partial\Omega}|z-x_0|$ and let $\nu_y$ be the inward pointing unit normal of $\Omega$ at $y$. For the function

$$\varphi_y:\mathbb{R}^n\to\mathbb{R},\quad \varphi_y(x):=\langle\nu_y,x-y\rangle$$

it then follows

$$\varphi(x_0)>u_\varepsilon(x_0). \tag{2.31}$$

Observe that we have

$$Q_\varepsilon(\varphi_y)=0\quad\text{on }\Omega.$$

for the quasilinear differential operator

$$Q_\varepsilon(u):=\varepsilon\Delta u-|Du|^2+1,\quad u\in C^2(\Omega).$$

As $\Omega$ is convex, we furthermore have $\varphi_y \geq 0$ on $\partial\Omega$. On the other hand, we have $Q_\varepsilon(u_\varepsilon) = 0$ on $\Omega$ as well as $u_\varepsilon \equiv 0$ on $\partial\Omega$. Then the quasilinear maximum principle (cf. [9]) implies $u_\varepsilon \leq \varphi_y$ on $\bar{\Omega}$, a contradiction to (2.31).

Similarly, the quasilinear maximum principle implies that we have $u_\varepsilon \geq 0$ on $\bar{\Omega}$, whence altogether it follows

$$0 \leq u_\varepsilon \leq d \quad \text{on } \bar{\Omega}.$$

As $u_\varepsilon \in C^2(\bar{\Omega})$, this implies

$$\left| \frac{\partial}{\partial v} u_\varepsilon(x) \right| \leq 1 \tag{2.32}$$

for all $x \in \partial\Omega$ and for any direction $v \in \mathbb{R}^n$, $|v| = 1$. Now fix $v \in \mathbb{R}^n$ with $|v| = 1$. Differentiating (2.23) with respect to $v$ yields

$$\varepsilon \Delta w - 2\langle Du_\varepsilon, Dw \rangle = 0$$

where $w := \frac{\partial}{\partial v} u_\varepsilon$. By (2.32) the linear maximum principle then implies $|w| \leq 1$ on $\bar{\Omega}$. As the choice of $v$ was arbitrary, it follows $|Du_\varepsilon| \leq 1$ on $\bar{\Omega}$. Hence

$$\lim_{\varepsilon \to 0} \int_\Omega \left| |Du_\varepsilon|^2 - 1 \right| \, dx = - \lim_{\varepsilon \to 0} \int_\Omega |Du_\varepsilon|^2 - 1 \, dx = 0$$

by lemma 2.6. This completes the proof.

□

## 2.4 Generalized solutions in the sense of Kružkov

In view of the historical development of the concept of viscosity solutions, it is worth while elaborating on the work of S. N. Kružkov, especially on his achievements in developing a comprising theory of generalized solutions of "Hamilton-Jacobi equations of eikonal type". The corresponding paper [5] provided valuable inspiration for the present thesis, and we therefore outline the essential ideas of his theory in order to draw a complete picture.

Kružkov's theory applies to Hamilton-Jacobi equations of the form

$$H(Du, u, x) = 0 \quad \text{with} \quad H(Du, u, x) = f(Du, u, x) - n^2(u, x), \tag{2.33}$$

where we have $x \in \Omega$ for some domain $\Omega \subset \mathbb{R}^n$. The functions $n$ and $f$ are subject to several constraints, among which the most important are

$$\begin{array}{ll} (i) & f(0, u, x) = 0 \quad \text{for all } (u, x) \in \mathbb{R} \times \Omega \\ (ii) & H(p, u, x) \text{ is convex with respect to } p \\ (iii) & H(p, u, x) \text{ is nonincreasing with respect to } u. \end{array} \tag{2.34}$$

These equations are related to the eikonal equation, whereby in the geometrical-optics interpretation the function $n(u,x)$ corresponds to the index of refraction of light rays determined by the properties of the medium.

Classically, nonlinear problems like this can be attacked by the method of characteristics, usually yielding a unique classical solution in the vicinity of a given manifold of dimension $n-1$, provided that this manifold is noncharacteristic and sufficiently smooth. However, in the case of the above eikonal type equations, unicity cannot be expected for these local classical solutions, as the original equation may split into two distinct equations, each of them corresponding to a different local solution. Apart from that, the projections of the characteristics onto the underlying space intersect in general, with the consequence that at a point of intersection each of them corresponds to a different value of the gradient brought to it from the initial manifold. With this in mind it suggests itself to dispense with classical solutions and to allow for weak solutions satisfying the equation only almost everywhere. Then unicity is not guaranteed in the first stage – even in case of problems where the method of characteristics yields global solutions –, and one has to introduce an extra condition. Kružkov's condition can be most easily outlined in the case of the eikonal equation on an interval, that is

$$\begin{cases} |u_x| = 1 \quad \text{on } [0,1] \\ u(0) = u(1) = 0 \end{cases} \tag{2.35}$$

In the class of Lipschitz continuous functions satisfying (2.35) almost everywhere, the most obvious are those piecewise linear functions which are composed by sections of lines of slope 1 or $-1$ and which satisfy the boundary constraints. Among these solutions the distance function seems to be the most natural, as it possesses several extremal properties: It maximizes the volume functional, it minimizes a curvature functional (suitably weakly defined), and it is concave. In fact, if any of these extra conditions is demanded, the distance function will be uniquely singled out. However, only (a modification of) the concavity constraint turns out to be powerful enough to still provide uniqueness if the boundary conditions are more general, the dimension of the space is higher, or if the equation itself deviates from the eikonal equation in a sense which is referred to as "eikonal-type" by Kružkov. Whereas both volume and curvature functionals in connection to the eikonal equation will be discussed later on, let us at this stage elaborate on the modified concavity condition that has been introduced by Kružkov. In fact it is sufficient to ensure that the weakly defined second derivatives are bounded by above.

**Definition 2.1** *A function $u : \Omega \to \mathbb{R}$ is called a* generalized solution of *2.33* in the sense of Kružkov *, if it has the following properties:*

*(i)* *u is locally Lipschitz continuous on* $\Omega$

*(ii)* *u satisfies 2.33 almost everywhere*

*(iii)* *for each* $r > 0$ *and* $x \in \Omega$ *such that* $B_r(x) \subseteq \Omega$ *we have*

$$\frac{\Delta^2 u}{|\Delta x|^2} := \frac{u(x+\Delta x) - 2u(x) + u(x-\Delta x)}{|\Delta x|^2} \leq C(x,r)$$

*for all* $\Delta x \in \mathbb{R}^n$ *with* $0 < |\Delta x| \leq r$.

Kružkov showed that if such a generalized solution exists, it will be unique. The proof is essentially based on property (iii) in combination with a certain transformation of the problem similar to the one we employed in the proofs of lemmas 2.1, 2.3, and 2.6. Furthermore, he applied the method of vanishing viscosity to obtain existence of his generalized solutions, which reflects the connection to the concept of viscosity solutions to be developed by Crandall and Lions some years later. In fact, Kružkov's theory coincides with their theory in the case of eikonal-type equations. All this has been exhaustively discussed in the literature.

## 2.5 Viscosity solutions in the sense of Crandall and Lions

Michael G. Crandall and Pierre-Louis Lions were the first to notice that Kružkov's additional constraint – the local upper bound of the second derivatives – can be seen as the manifestation in a special case of a by far more elegant and appealing concept applicable to a broad class of Hamilton-Jacobi equations, which led to the concept of viscosity solutions. Since the fruitful research activities triggered by the first introduction of this concept has changed the original formulation dramatically, we will first present the historically oldest definition, which captures the essential features in the best way. We also restrict ourselves to problems of the form

$$H(Du, u, x) = 0 \quad \text{in } \Omega, \tag{2.36}$$

where $\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, $H : \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$. Moreover we demand the fundamental monotonicity condition

$$H(p,r,x) \leq H(p,s,x) \quad \text{whenever } r \leq s \text{ for all } (p,x) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.37}$$

which is essential for the theory (cf. [6]).

We now provide the important notion of upper and lower test functions.

**Definition 2.2** *Let* $u \in C(\Omega)$ *and let* $x \in \Omega$. *A function* $\varphi \in C(\Omega)$ *which is differentiable at* $x$ *and for which* $u - \varphi$ *attains a local maximum (minimum) at* $x$ *is called* upper (lower) test function of $u$ at $x$.

A viscosity solution of (2.36) is then defined via a test function condition according to the following

**Definition 2.3** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $x \in \Omega$. For problem (2.36), a continuous function $u : \Omega \to \mathbb{R}$ is said to satisfy the* viscosity subsolution, viscosity supersolution, or viscosity solution condition at $x$, *if respectively the first, the second, or both of the following conditions are satisfied.*

*(i) For all upper test functions $\varphi$ of $u$ at $x$ we have $H(D\varphi(x), u(x), x) \leq 0$.*

*(ii) For all lower test functions $\varphi$ of $u$ at $x$ we have $H(D\varphi(x), u(x), x) \geq 0$.*

*If $u$ satisfies the viscosity subsolution, viscosity supersolution, or viscosity solution condition for all $x \in \Omega$, then $u$ is respectively called a* viscosity subsolution, viscosity supersolution, or viscosity solution *of (2.36).*

As a matter of fact, this definition of viscosity solution allows the derivation of a whole bunch of existence and uniqueness results for boundary value problems which can be proven in a far more elegant way compared with Kružkov's methods (cf. [6]). It can also be shown that the vanishing viscosity limit coincides with the unique viscosity solution, a fact which we will refer to as *consistency with the method of vanishing viscosity*.

# Chapter 3

# Perron methods for the eikonal equation

**Summary.** The present chapter relates viscosity solutions of the eikonal equation on non-ramified domains to maximum volume solutions of granular matter problems. We show the equivalence of the two selection principles induced by the test function and the maximum volume constraint. In fact, we characterize viscosity solutions of the viscous and non-viscous eikonal equation as pointwise suprema over a certain class of subsolutions, the so-called *subeikonal* equations, respectively (*Perron method*). Similary, we consider the viscous version of the eikonal equation and characterize its viscosity solutions as suprema of *subharmoneikonal* functions.

## 3.1 Introduction

In [1], the authors discuss the analogy between the Dirichlet problem of the Laplacian and the Dirichlet problem of the eikonal equation under a maximal volume constraint on bounded domains. Among other things, they point out that in both cases one can apply a Perron method to construct solutions. Whereas the Perron method for the Laplacian is well known, for the maximal value problem they introduce classes of subfunctions (the so-called *subeikonal* functions), and show that solutions are given by pointwise suprema of all subeikonal functions staying below given boundary values.

On the other hand, as has been shown by Ishii in [11], a Perron method can be designed to construct viscosity solutions for a general class of Hamilton-Jacobi equations. In fact, a viscosity solution is given by the pointwise supremum over all viscosity subsolutions staying below given boundary values.

We demonstrate that Ishii's method applied to the eikonal equation can be replaced by the method given in [1]. As we will see, the class of subeikonal functions is contained in the class of viscosity subsolutions and has the property that its pointwise suprema as constructed in [1] are in fact viscosity solutions.

Secondly we consider the *viscous* version of the eikonal equation, which reads

$$\varepsilon \Delta u - |Du| + 1 = 0 \tag{3.1}$$

We propose a class of *subharmoneikonal functions* (*SHE functions*, for short), which is a modification of the class of subeikonal functions. We show once more that a Perron method based on these functions yields viscosity solutions. For this purpose we will have to invoke the definition of viscosity solutions extended to second order equations.

## 3.2 The eikonal equation and subeikonal functions

Throughout this chapter let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider the eikonal equation

$$|Du| - 1 = 0 \tag{3.2}$$

on $\Omega$. The following terminology is introduced in [1].

**Definition 3.1** *Let $x \in \Omega$. We call a function $u \in C(\bar{\Omega})$* subeikonal at *$x$, whenever there is a radius $r_0 > 0$ with $B_{r_0}(x) \subseteq \Omega$ such that*

$$u(x) \leq \inf_{y \in S_r(x)} u(y) + r \tag{3.3}$$

*for all $0 < r \leq r_0$, where $S_r(x) := \partial B_r(x)$. The function $u$ is called* subeikonal*, if it is subeikonal at each point $x \in \Omega$.*

Suppose the boundary of the domain $\Omega$ be Lipschitz. For a given boundary data function $\phi : \partial\Omega \to \mathbb{R}$ we define $X$ to be the set of all subeikonal functions $u$ with $u \leq \phi$ on $\partial\Omega$. It has been shown in [1] that the function $\tilde{u} : \bar{\Omega} \to \mathbb{R}$ defined by

$$\tilde{u}(x) := \sup_{u \in X} u(x), \quad x \in \bar{\Omega}, \tag{3.4}$$

is also contained in $X$ (consistency). Furthermore, $\tilde{u}$ is the unique function among all Lipschitz continuous functions with Lipschitz constant 1, which maximizes the volume functional

$$V(u) := \int_\Omega u(x)\, dx.$$

The construction of the maximum volume solution $\tilde{u}$ as a pointwise supremum of subeikonal functions is what we refer to as *Perron's method*. We now show that the *Perron solution* $\tilde{u}$ is in fact a viscosity solution of (3.2). We start with the following

**Lemma 3.1** *Let $v \in C(\bar{\Omega})$ be subeikonal. Then $v$ is a viscosity subsolution of (3.2).*

**Proof.** Let $x \in \Omega$ and let $\varphi$ be an upper test function of $v$ at $x$. We have to show $|D\varphi(x)| \leq 1$. First note that we can without loss of generality assume that we have $v(x) = \varphi(x)$. By the fact that $v$ is subeikonal and by the fact that $v - \varphi$ attains a local maximum at $x$ we conclude that there is a number $\tilde{r} > 0$ such that

$$\varphi(x) = v(x) \leq \inf_{y \in S_r(x)} v(y) + r \leq \inf_{y \in S_r(x)} \varphi(y) + r \tag{3.5}$$

for all $0 < r \leq \tilde{r}$.

As $\varphi$ is differentiable at $x$, we can write down its Taylor expansion

$$\varphi(x + re) = \varphi(x) + r\langle D\varphi(x), e\rangle + O(r^2)$$

for any $e \in S^{n-1}$. Choosing $e := -D\varphi(x)/|D\varphi(x)|$, we conclude

$$\inf_{y \in S_r(x)} \varphi(y) = \inf_{z \in S^{n-1}} \varphi(x + rz) \leq \varphi(x + re) = \varphi(x) - r|D\varphi(x)| + O(r^2). \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$0 \leq 1 - |D\varphi(x)| + O(r),$$

which implies $|D\varphi(x)| \leq 1$ upon letting $r \to 0$. □

It has been shown in [1] that $\tilde{u}$ is subeikonal. Hence the above lemma implies that $\tilde{u}$ is a viscosity subsolution of (3.2). It remains to show that it also is a viscosity supersolution.

**Lemma 3.2** *The function $\tilde{u}$ as defined in (3.4) is a viscosity supersolution of (3.2).*

**Proof.** Suppose the contrary were the case. Then there is a point $x_0 \in \Omega$ and a lower test function $\varphi$ of $\tilde{u}$ at $x_0$ such that we have $|D\varphi(x_0)| < 1$. Without loss of generality we may assume $\tilde{u}(x_0) = \varphi(x_0)$ and that $\varphi$ be $C^2$ in an open neighbourhood $U$ of $x_0$. Furthermore we can assume that the local minimum of $u - \varphi$, which by definition is attained at $x_0$, be *strict*, by possibly adding to $\varphi$ a paraboloid of the form

$$y \mapsto -\alpha||x_0 - y||^2, \quad \alpha > 0.$$

For reasons of continuity it follows that there are small numbers $\eta, \xi > 0$ such that $\bar{B}_\xi(x_0) \subset U$ and such that for $\tilde{\varphi} := \varphi + \eta$ we have $\tilde{\varphi}(y) < u(y)$ for all $y \in \partial B_\xi(x_0)$ as well as $|D\tilde{\varphi}(y)| < 1$ for all $y \in B_\xi(x_0)$. We then show that the function $\bar{u} \in C(\bar{\Omega})$ given by

$$\bar{u}(x) := \begin{cases} \max\{\tilde{u}(x), \tilde{\varphi}(x)\} & \text{if } x \in \bar{B}_{\xi/2}(x_0) \\ \tilde{u}(x) & \text{if } x \in \bar{\Omega}\backslash\bar{B}_{\xi/2}(x_0) \end{cases}$$

is containe in $X$. For this purpose first note that $\bar{u}$ clearly is subeikonal for all $x \in \Omega \backslash \bar{B}_{\xi/2}(x_0)$. Now let $x \in \bar{B}_{\xi/2}(x_0)$. As $\varphi$ is $C^2$ in $U$ and as $x \in U$, there is a radius $r_0 > 0$ such that by Taylor expansion we have

$$\begin{aligned} \varphi(x+re) &> \varphi(x) + r\langle D\varphi(x), e\rangle - Cr^2 \geq \varphi(x) - r\left\langle D\varphi(x), \frac{D\varphi(x)}{|D\varphi(x)|}\right\rangle - Cr^2 \\ &= \varphi(x) - r|D\varphi(x)| - Cr^2 \end{aligned}$$

for all $e \in S^{n-1}$ and all $0 < r \leq r_0$. Here $C > 0$ is a constant independent of $r$ and $e$. Consequently, as $|D\varphi(x)| < 1$, there is a $\delta > 0$ such that for all $e \in S^{n-1}$ and all $0 < r \leq r_0$ we have

$$\varphi(x+re) + r > \varphi(x) + \delta r - Cr^2.$$

Hence for all $0 < r < \delta/C := r_1$ we have

$$\inf_{y \in S_r(x)} u(y) + r = \inf_{e \in S^{n-1}} u(x+re) + r > \varphi(x). \tag{3.7}$$

On the other hand, $\tilde{u}$ is subeikonal at $x$ according to [1], whence there is an $r_2 > 0$ such that we have

$$\tilde{u}(x) \leq \inf_{y \in S_r(x)} \tilde{u}(y) + r \tag{3.8}$$

for all $0 < r < r_2$. Combining (3.7) and (3.8) we find

$$\max\{\tilde{u}(x), \tilde{\varphi}(x)\} \leq \inf_{y \in S_r(x)} \max\{\tilde{u}(y), \tilde{\varphi}(y)\} + r$$

for all $0 < r < \min\{r_1, r_2\}$, implying that $\bar{u}$ is subeikonal at $x$. Thus we have $\bar{u} \in X$. Since by construction we have $\bar{u}(x_0) > \tilde{u}(x_0)$, we obtain a contradiction to the definition of $\tilde{u}$. This completes the proof. $\square$

The combination of lemmas 3.1 and 3.2 yields the following

**Corollary 3.1** *The function $\tilde{u}$ as defined in (3.4) is a viscosity solution of (3.2).*

As the above lemmas have shown, the Perron method designed in [1] yields a viscosity solution of the eikonal equation. Note that the behaviour of both the boundary $\partial\Omega$ and the boundary data $\phi$ is not relevant in the above proofs. However, as it is also the case for the classical Perron method for the Dirichlet problem, it is not clear if the boundary data are assumed. We thus need an extra *barrier* condition for the boundary values. Such a barrier condition is given in [1].

## 3.3 The viscous eikonal equation and SHE functions

We have seen that in the simple case of the eikonal equation viscosity solutions can be constructed by means of a Perron method, that is, as pointwise suprema of subeikonal functions. The subeikonal functions are special, easy-to-describe viscosity subsolutions, and are intuitively related to the eikonal equation itself.

We now modify the eikonal equation by adding a viscosity term and pose the question, if solutions of the resulting *viscous eikonal equation*, which is a semilinear second order equation, can be characterized by a similarly intuitive class of "subsolutions".

For the remainder of this chapter let $\Omega \subset \mathbb{R}^n$ be a bounded domain and fix $\varepsilon > 0$. For $u \in C^2(\Omega)$ we define the semilinear operator $Q$ by

$$Q(u) := \varepsilon \triangle u - |Du| + 1.$$

The equation

$$Q(u) = \varepsilon \triangle u - |Du| + 1 = 0 \tag{3.9}$$

is called *viscous eikonal equation*, where we emphasize that – in contrast to the previous chapter – the term $|Du|$ is not squared here.

Our aim is to show that if a classical solution of (3.9) exists, then it can be characterized as the pointwise supremum over a suitable class of "subfunctions". In a way, equation (3.9) is a linear combination of the Laplace equation and the eikonal equation, and the corresponding subfunctions are expected to reflect properties of each of them. In fact, subeikonal solutions can be described by a local "subextremal" property, whereas subharmonic functions satisfy a local "subaverage" condition. Note how both of these properties are maintained in the following definition.

**Definition 3.2** *Let $x \in \Omega$. We call a function $u \in C(\Omega)$* subharmoneikonal (SHE) at *$x$, if there is an $r_0 > 0$ such that $B_{r_0}(x) \subseteq \Omega$ and such that for all $0 < r \leq r_0$ we have*

$$u(x) \leq \frac{r}{r + 2n\varepsilon} \left( \inf_{y \in S_r(x)} u(y) + r \right) + \frac{2n\varepsilon}{r + 2n\varepsilon} \oint_{S_r(x)} u(y)\, dS(y), \tag{3.10}$$

*where $\oint$ denotes the average integral.*

*Furthermore let $A \subseteq \Omega$. If $u$ is SHE at all $x \in A$, we say that $u$ is subharmoneikonal (SHE) on $A$ and write $u \in SHE(A)$.*

**Definition 3.3** *Let $u, v \in C(\Omega)$ and $x \in \Omega$. We say that $v$* touches $u$ by above (by below) *at $x$, if $u(x) = v(x)$ and if there is a neighbourhood $U$ around $x$ such that $v(y) \geq$ $(\leq)$ $u(y)$ for all $y \in U$.*

The following two propositions are immediate consequences of definition 3.2.

**Proposition 3.1** *Let $u \in C(\Omega)$ be SHE at $x \in \Omega$ and assume that $v \in C(\Omega)$ touches $u$ by above at $x$. Then $v$ is SHE at $x$.*

**Proposition 3.2** *Let $u, v \in C(\Omega)$ be SHE at $x \in \Omega$. Then $w := \max\{u, v\}$ is SHE at $x$.*

**Lemma 3.3** *Let $x \in \Omega$ and let $u : \Omega \to \mathbb{R}$ be twice differentiable at $x$.*
*(i) If $u$ is SHE at $x$, then $Q(u)(x) \geq 0$.*
*(ii) If $Q(u)(x) > 0$, then $u$ is SHE at $x$.*

**Proof.** Expanding $u$ around $x$ yields

$$u(x + re) = u(x) + r\langle Du(x), e\rangle + \frac{r^2}{2} e^t D^2 u(x) e + O(r^3), \tag{3.11}$$

for any $e \in S^{n-1}$. We integrate and obtain

$$\fint_{y \in S_r(x)} u(y)\, dS(y) = \fint_{e \in S^{n-1}} u(x + re)\, dS(e) = u(x) + \frac{r^2}{2n} \Delta u(x) + O(r^3), \tag{3.12}$$

where we have used the relations

$$\int_{e \in S^{n-1}} \langle Du(x), e\rangle\, dS(e) = 0$$

and

$$\int_{e \in S^{n-1}} e^t A e\ dS(e) = \frac{1}{n} \cdot |S^{n-1}| \cdot \mathrm{tr} A$$

for any symmetric $n \times n$-matrix $A$.

Moreover, by setting $e = -Du/|Du|$ we obtain the relation

$$\inf_{y \in K_r(x)} u(y) \leq u(x) - r|Du(x)| + O(r^2). \tag{3.13}$$

(i) Let $u$ be SHE at $x$. By (3.10), (3.12) and (3.13) we estimate

$$\begin{aligned} 0 &\leq r \left[ \inf_{y \in K_r(x)} u(y) + r - u(x) \right] + 2n\varepsilon \left[ \fint u(y)\, dS(y) - u(x) \right] \\ &\leq -r^2 |Du(x)| + r^2 \varepsilon \Delta u(x) + r^2 + O(r^3). \end{aligned}$$

Dividing by $r^2$ and letting $r \to 0$ yields $Q(u)(x) \geq 0$.

(ii) Conversely, suppose that we have $Q(u)(x) > 0$. It follows that there is a number $\delta > 0$ such that $Q(u)(x) - \delta > 0$. Plugging $e = -Du/|Du|$ into (3.11) and multiplying by $r$ yields that for all sufficiently small $r > 0$ we have

$$r \inf_{y \in K_r(x)} u(y) \geq ru(x) - r^2|Du(x)| + O(r^3) - r^2\delta/2. \tag{3.14}$$

Hence we have by (3.13) and (3.14)

$$\begin{aligned} 0 &< r^2\varepsilon\Delta u(x) - r^2|Du(x)| + r^2 - r^2\delta \\ &\leq r\left[\inf_{y \in K_r(x)} u(y) + r - u(x)\right] + 2n\varepsilon\left[\oint u(y)\,dS(y) - u(x)\right] - r^2\delta/2 + O(r^3) \\ &\leq r\left[\inf_{y \in K_r(x)} u(y) + r - u(x)\right] + 2n\varepsilon\left[\oint u(y)\,dS(y) - u(x)\right] \end{aligned}$$

for all sufficiently small $r > 0$, implying that $u$ is SHE at $x$. □

Now we generalize the technical procedure applied to the eikonal equation in the previous section. For this purpose we extend the concept of viscosity solutions to second order equations. There are different ways to accomplish this; the easiest is to simply replace the $C^1$ test functions in definition 2.3 by $C^2$ test functions. We remark, however, that viscosity solutions of second order will not appear in the subsequent chapters, which is why we dispense with further details. For the general theory we refer to [6].

**Definition 3.4** *A function $u \in C(\Omega)$ is said to be a* viscosity subsolution (supersolution) *of*

$$Q(u) = 0, \tag{3.15}$$

*if the following holds: For any $\varphi \in C^2(\Omega)$ and any $y \in \Omega$ such that $u - \varphi$ has a local maximum (minimum) at $y$ we have $Q(\varphi)(y) \geq (\leq)\ 0$.*

*The function $u$ is called* viscosity solution of (3.15)*, if it is both a sub- and a supersolution of (3.15). We denote the class of all viscosity subsolutions, supersolutions, solutions of (3.15) on $\Omega$ by $\underline{S}(\Omega)$, $\bar{S}(\Omega)$, and $S(\Omega)$, respectively.*

**Lemma 3.4** *We have $SHE(\Omega) \subset \underline{S}(\Omega)$.*

**Proof.** Let $u \in \mathrm{SHE}(\Omega)$, $\varphi \in C^2(\Omega)$ and let $u - \varphi$ attain a local maximum at some point $y \in \Omega$. If we also assume that $u(y) = \varphi(y)$, it follows that $\varphi$ touches $u$ by above. According to proposition 3.1, $\varphi$ is SHE at $y$, implying $Q(\varphi)(y) \geq 0$ by lemma 3.3. Hence we have $u \in \underline{S}(\Omega)$. □

Now we fix any continuous boundary data function $\phi : \partial\Omega \to \mathbb{R}$ and define the set

$$X := \{u \in \mathrm{SHE}(\Omega) \,|\, u \leq \phi \text{ on } \partial\Omega\}.$$

As before we define the Perron solution $\bar{u}$ to be the pointwise supremum over $X$, i.e.

$$\bar{u}(x) := \sup_{u \in X} u(x),$$

and show that it is a viscosity solution of (3.15).

**Lemma 3.5** *We have $\bar{u}(x) < \infty$ for all $x \in \Omega$.*

**Proof.** As $\partial\Omega$ is compact and as $\phi$ is continuous, there is an $m > 0$ such that $\phi \leq m$ on $\partial\Omega$. Consequently, $u \leq m$ on $\partial\Omega$ for all $u \in X$.

Choose a point $x \in \Omega$ and set $d := \sup\{||x - z||, z \in \partial\Omega\} > 0$. Define the function

$$v : \Omega \to \mathbb{R}, \quad v(y) := -\alpha||y - x||^2 + K,$$

where $K > \alpha d^2 + m$ and $\alpha > 0$ is yet to be chosen. It follows that $v > m$ on $\partial\Omega$. Furthermore we have

$$Q(v)(y) = -2\varepsilon n\alpha - 2\alpha||y - x|| + 1.$$

If we now choose $\alpha > (2\varepsilon n)^{-1}$, we have $Q(v) < 0$ in $\Omega$. Hence $v$ is a classical supersolution of $Q(u) = 0$. Thus it is also a viscosity supersolution. Let $u \in X$. By lemma 3.4 it follows that $u$ is a viscosity subsolution of $Q(u) = 0$. As $u < v$ on $\partial\Omega$ by construction, we invoke the comparison theorem for viscosity solutions (theorem 3.3 in [6]) to conclude that $u \leq v$ in $\Omega$. We obtain that $\bar{u} \leq v$ in $\Omega$. The assertion follows. $\square$

**Lemma 3.6** *The function $\bar{u}$ is a viscosity supersolution of $Q(u) = 0$.*

**Proof.** Assume the contrary. Then there is a point $y \in \Omega$ and a function $\varphi \in C^2(\Omega)$, such that $\bar{u} - \varphi$ has a local minimum at $y$ and $Q(\varphi)(y) > 0$. We may assume $\bar{u}(y) = \varphi(y)$.

Now choose $\delta > 0$ small enough to have $Q(\varphi)(y) - 2\varepsilon\delta > 0$ and define the paraboloid

$$\psi(x) := \varphi(y) + D\varphi(y)(x - y) + \frac{1}{2}(x - y)^t D^2\varphi(y)(x - y) - \frac{\delta}{2}||x - y||^2.$$

Then we have $\triangle\psi(y) = \triangle\varphi(y) - 2\delta$, whence $Q(\psi)(y) > 0$. Moreover, there is a radius $s > 0$ such that $\bar{B}_s(y) \subset \Omega$ and $\psi < \varphi$ on $B_s(y)\backslash\{y\}$. Continuity implies that there is a radius $t$ with $s \geq t > 0$ such that $Q(\psi) > 0$ on $\bar{B}_t(y)$. It follows that $\psi \in \mathrm{SHE}(\bar{B}_t(y))$ according to lemma 3.3 (ii).

Let $\xi := \min_{\partial B_t(y)}(\varphi - \psi)$ and observe that we have $\xi > 0$. Define $\bar{\psi} := \psi + \frac{\xi}{3}$. Clearly $\bar{\psi}(y) > \bar{u}(y)$. Furthermore we have $\bar{\psi} \in \mathrm{SHE}(\bar{B}_t(y))$.

Now note that we have $\bar{u} \geq \psi + \xi$ on $\partial B_t(y)$. Hence for each $x \in \partial B_t(y)$ there is a function $v \in X$ such that $v(x) > \psi(x) + \frac{2\xi}{3}$. Due to the continuity of $y$ and $\psi$ and the compactness of $\partial B_t(y)$ there are finitely many $v_i \in X$, $1 \leq i \leq m$, such that $v := \max v_i > \psi + \frac{2\xi}{3}$ on $\partial B_t(y)$. We have $v \in X$ by proposition 3.2.

By reasons of continuity it follows that for each $x \in \partial B_t(y)$ we have $w \equiv v$ on a neighbourhood of $x$, where $w$ is defined by

$$w := \begin{cases} \max\{\bar{\psi}, v\} & \text{on } \bar{B}_t(y) \\ v & \text{on } \Omega \backslash \bar{B}_t(y). \end{cases}$$

We conclude that $w$ is SHE at each $x \in \partial B_t(y)$. Furthermore, by proposition 3.2 we have $w \in \mathrm{SHE}(B_t(y))$, as $v, \bar{\psi} \in \mathrm{SHE}(B_t(y))$. Clearly, $w \in \mathrm{SHE}(\Omega \backslash \bar{B}_t(y))$. Furthermore we have $w \in C(\Omega)$. Hence it follows that $w \in \mathrm{SHE}(\Omega)$ and also $w \in X$, since $v \equiv w$ on $\partial\Omega$. This, however, is a contradiction to the supremal property of $\bar{u}$, as $w(y) > \bar{u}(y)$. □

It remains to show that $\bar{u}$ is a viscosity subsolution. For this purpose we invoke an alternative definition of viscosity (sub-, super-) solutions of second order equations. The idea is to replace upper and lower test functions by super- and subdifferentials (or *semi-jets*) according to the following

**Definition 3.5** *Let $u : \Omega \to \mathbb{R}$ be locally bounded and $x \in \Omega$. The* super- (sub-) differential $D^+u(x)$ $(D^-u(x))$ of $u$ at $x$ *is the set of all pairs $(p, S)$ of a vector $p \in \mathbb{R}^n$ and a symmetric $n \times n$ matrix $S$ such that*

$$u(y) \leq \ (\geq)\ u(x) + \langle p, y - x\rangle + \frac{1}{2}(y - x)^t S(y - x) + o(||y - x||^2)$$

*as $y \to x$.*

The possibility to characterize viscosity solutions by means of semi-jets instead of test functions is confirmed by the following

**Lemma 3.7** *We have $u \in \underline{S}$ $(\in \bar{S})$, if and only if*

$$\tilde{Q}(p, S) := \varepsilon trS + 1 - |p| \geq \ (\leq)\ 0 \text{ for all } x \in \Omega \text{ and } (p, S) \in D^+u(x)\ (\in D^-u(x)).$$

**Proof.** Confer [6], for example. □

**Lemma 3.8** *Let $\mathcal{S} \subset SHE(\Omega)$ be an arbitrary set of SHE-functions and set $u(x) := \sup_{v \in \mathcal{S}} v(x)$ for all $x \in \Omega$. Assume $u(x) < \infty$ for all $x \in \bar{\Omega}$. Then $u \in \underline{S}$.*

**Proof.** Let $x \in \Omega$ and $(p, S) \in D^+u(x)$. We show $\tilde{Q}(p, S) \geq 0$. To this end define the paraboloid

$$P : \mathbb{R}^n \to \mathbb{R}, \quad P(y) := \langle p, y\rangle + \langle y, Sy\rangle,$$

and observe that from the definition 3.5 it follows that for any $\delta > 0$ there is a radius $r > 0$ such that

$$u(y) \leq u(x) + P(y - x) + \delta||y - x||^2 \text{ for all } y \in \bar{B}_r(x). \tag{3.16}$$

For any $k \in \mathbb{R}$ we now choose a function $u_k \in \mathcal{S}$ such that $u(x) - u_k(x) < 1/k$. Let $y_k$ be a point where the function

$$f_k : y \mapsto u_k(y) - P(y - x) - 2\delta||y - x||^2$$

attains its maximum in the compact set $\bar{B}_r(x)$.

Due to compactness, we may (by choosing a subsequence) assume that $y_k$ is convergent. In fact, we can show that $y_k \to x$. For this purpose we observe that since $y_k$ is a maximum point of $f_k$, we have

$$u_k(x) \leq u_k(y_k) - P(y - x) - 2\delta||y - x||^2,$$

and, as $u_k(y_k) \leq u(y_k)$ and $u_k(x) > u(x) - 1/k$,

$$u(x) - \frac{1}{k} < u(y_k) - P(y - x) - 2\delta||y - x||^2. \tag{3.17}$$

On the other hand, relation (3.16) yields

$$u(y_k) \leq u(x) + P(y_k - x) + \delta||y_k - x||^2. \tag{3.18}$$

If we add (3.17) and (3.18), we get $1/k > \delta||y_k - x||^2$, whence $y_k \to x$ as $k \to \infty$.

Consequently, for sufficiently large $k \in \mathbb{N}$, we have $y_k \in B_r(x)$. It follows that the function

$$y \mapsto u_k(y) - P(y - x) - 2\delta||y - x||^2$$

attains an (interior) local maximum at $y_k$ for all large $k$. Therefore, setting

$$\varphi_\delta(y) := P(y - x) + 2\delta||y - x||^2,$$

we have $Q(\varphi_\delta)(y_k) \geq 0$, since $u_k$ is SHE at $y_k$ and thus also is a viscosity subsolution due to lemma 3.4. Now observe that we have

$$Q(\varphi_\delta) = \tilde{Q}(p, S) + 4n\varepsilon\delta,$$

whence $-4n\varepsilon\delta \leq \tilde{Q}(p, S)$. Since the choice of $\delta > 0$ was arbitrary, we conclude $\tilde{Q}(p, S) \geq 0$. □

**Theorem 3.1** *The function $\bar{u}$ is a viscosity solution of $Q(u) = 0$ in $\Omega$.*

**Proof.** Combine lemma 3.6 and lemma 3.8, where in the latter we set $\mathcal{S} = X$, and invoke lemma 3.5. □

# Chapter 4

# Vanishing viscosity on networks

**Summary.** In this chapter we generally introduce the notion of ramified spaces, and, in particular, networks. We apply the vanishing viscosity method to first order Hamilton-Jacobi equations on networks. First we treat the case of the eikonal equation, followed by a convergence result for a general class of Hamilton-Jacobi equations on networks.

## 4.1 Introduction

The concept of ramified spaces has originally been introduced by Gunter Lumer [12] and has later been refined and specified by various authors, e.g. J. v. Below and S. Nicaise [13]. It serves as an appropriate setting for problems of interaction between different media, governed by partial differential equations on the branches and transition conditions on the ramification spaces. Interaction problems find various applications in physics, chemistry, and biology – confer for example [14] and [15]. Also, well-known models based on scalar equations appear in a new light when being embedded into the ramified space setting, such as the description of the behaviour of chemical substances by reaction-diffusion equations [16]. As far as the analysis of these models is concerned, the applicability of various mathematical methods does not only depend on the structure of the differential equations on the branches, but also and particularly on the properties of the transition conditions. In fact, the latter have an considerable effect on existence, uniqueness, and regularity of solutions. Possible aspects regarding which the transition conditions may vary are linearity or nonlinearity, being dynamical or static, dissipative or nondissipative, etc.

Many well-known elliptic and parabolic concepts have been adapted from the classical non-ramified situation to ramified spaces, providing results for boundary or initial value problems, formulated for families of media with transition conditions. In many cases, the one-dimensional version of a ramified space, the so-called *topological network*, is of major importance, as the core of the problems often arises fully in this 'simple' setting. Also, higher dimensional problems sometimes come along with considerable technical difficulties.

However, as far as we know, research has been restricted to linear and semilinear elliptic (stationary) equations [17] on the one hand, and, on the other hand, to evolution equations such as (nonlinear) scalar reaction-diffusion equations [16]. Typically, the first demand adaption of Sobolev space methods and the theory of elliptic operators, whereas the latter are usually attacked by means of semigroup theory and other functional analytical tools such as fixed point theorems. However, fully nonlinear stationary problems on ramified spaces have not enjoyed similar attention, and according to our knowledge there has not been done any work in this area so far.

As in this thesis it is our goal to extend the theory of viscosity solutions to ramified spaces, we consider it natural to start with an corresponding extension of the method of vanishing viscosity. Again we begin with the eikonal equation by applying vanishing viscosity to a Dirichlet problem of the eikonal equation on networks. Two important aspects are discussed: The necessity of introducing an extra condition at transition vertex points for the viscous eikonal equation, as well as the question of convergence. Then we consider a general class of Hamilton-Jacobi equations, an important feature of which is to be "eikonal-type" at the transition vertices, and show that the vanishing viscosity method converges.

The fact that the extended method of vanishing viscosity converges motivates the extension of the theory of viscosity solutions for this class of equations, as the two concepts should coincide. In particular, observing the behaviour of the converging family at transition vertices should give hints to how to formulate a natural transition condition for viscosity solutions.

## 4.2 Ramified spaces

Let us start with the general definition of ramified spaces originally given by Lumer [12].

**Definition 4.1** *Let $\Omega^\star$ be a non-empty, separable, locally compact space with a countable topological basis. Let $\mathcal{L} = \{\Omega_i\}_{i\in I}$ be a countable family of non-empty open subsets $\Omega_i$ of $\Omega^\star$. Furthermore let $N_e^\star$ be a closed (possibly empty) subset of the set $N^\star := \Omega^\star \setminus \cup_{i\in I} \Omega_i$ with the property that it contains each point of $N^\star$ which is contained in the boundary of exactly one $\Omega_i$, $i \in I$.*

*We call $\Omega := \Omega^\star \setminus N_e^\star$ a ramified space, whenever we have*

$$\begin{array}{ll} (i) & \bar{\Omega}_i \cap \bar{\Omega}_j \subseteq \partial\Omega_i \cap \partial\Omega_j \quad \text{for } i,j \in I,\ i \neq j \\ (ii) & \Omega^\star = \cup_{i\in I}\bar{\Omega}_i \\ (iii) & \{\Omega_i\}_{i\in I} \text{ is locally finite in } \Omega^\star \\ (iv) & \Omega \text{ is connected.} \end{array}$$

$N_r := N^\star \backslash N_e^\star$ *is then called* ramification space *of* $\Omega$.

**Remark.** Observe that $\Omega$ is locally compact, as $N_e^\star$ is closed.

As the purely topological nature of this definition is too general for our purposes, we now introduce topological networks as important instances of ramified spaces. More general ramified spaces and examples will be discussed in chapter 7.

## 4.3 Graphs, topological graphs, and networks

In the following we distinguish between (abstract) graphs, topological graphs, and networks. We start with abstract graphs.

**Definition 4.2** *An (abstract)* graph $G$ *is a pairing* $G = (V, E)$, *where* $V = V(G)$ *and* $E = E(G)$ *are the sets of* vertices *and* edges, *respectively. An edge* $e \in E$ *is an unordered pair* $\{v_1, v_2\}$ *of vertices* $v_1, v_2 \in V$; *we write* $e = v_1 v_2$. *A* path *in* $G$ *is a formal sequence* $v_1 v_2 \dots v_n$ *of vertices* $v_1, \dots, v_n \in V$, $n \in \mathbb{N}$, *such that* $v_i v_{i+1} \in E$ *for all* $i = 1, \dots, n-1$.

We also provide the basic graph theoretical notions we will require in the sequel.

**Definition 4.3** *Let* $G = G(V, E)$ *be a graph.*

- *(i)* *We say that two vertices* $v, w \in V$ *are* adjacent, *whenever* $vw \in E$. *We write* $v \, adj \, w$.
- *(ii)* *We say that a vertex* $v \in V$ *and an edge* $e \in E$ *are* incident, *whenever there is a vertex* $w \in V$ *such that* $vw = e$. *We write* $v \, inc \, e$.
- *(iii)* $G$ *is called* connected, *if there is a path with endpoints* $v$ *and* $w$ *for each pair of vertices* $v, w \in V$.
- *(iv)* *For each vertex* $v \in V$ *we define its* degree $deg \, v$ *by* $deg \, v := |\{e \in E \mid v \, inc \, e\}|$.

In the sequel we assume each graph to be nonempty, finite, simple, and free of loops, which is expressed by the conditions

$$\begin{array}{ll} (i) & 0 < |V| < \infty \\ (ii) & |\{e \in E \mid e = vw\}| = 1 \quad \text{for all } v, w \in V,\ v \neq w \\ (iii) & v \neq w \text{ for all } e = vw \in E. \end{array}$$

For further graph theoretical terminology we refer to [18]. We now introduce the notion of a topological graph.

**Definition 4.4** *Let $V = \{v_i, i \in I\}$ be a finite collection of pairwise different points in $\mathbb{R}^n$. Furthermore, let $\{\pi_j, j \in J\}$ be a finite collection of continuous, non-self-intersecting curves in $\mathbb{R}^n$ given by*

$$\pi_j : [0, l_j] \to \mathbb{R}^n, \quad l_j > 0,\, j \in J.$$

*We set $e_j := \pi_j(\,]0, l_j[)$, $\bar{e}_j := \pi_j([0, l_j])$, and $E := \{e_j, j \in J\}$. Assume furthermore the following conditions to be satisfied:*

$$\begin{array}{rl} (i) & \pi_j(0), \pi_j(1) \in V \textit{ for all } j \in J \\ (ii) & |\bar{e}_j \cap V| = 2 \textit{ for all } j \in J \\ (iii) & \bar{e}_j \cap \bar{e}_h \subset V \textit{ and } |\bar{e}_j \cap \bar{e}_h| \le 1 \textit{ for all } j, h \in J,\ j \neq h. \end{array}$$

*Then $G := (V, E)$ is called a* (finite) topological graph.

Obviously we can consider $G$ not only as a subset of $\mathbb{R}^n$, but also as an abstract graph $G := (V, E)$ with vertices $V$ and edges $E$. As long as confusions are ruled out, we will interchange both interpretations without mentioning.

Observe that the parametrizations $\pi_j$ induce an orientation on the edges, which can be expressed by the *signed incidence matrix*

$$A = (a_{ij}) \quad \text{with} \quad a_{ij} := \begin{cases} 1 & \text{if} \quad v_i \operatorname{inc} e_j \quad \text{and} \quad \pi_j(0) = v_i \\ -1 & \text{if} \quad v_i \operatorname{inc} e_j \quad \text{and} \quad \pi_j(l_j) = v_i \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

**Definition 4.5** *Let $k \in \mathbb{N} \cup \{\infty\}$, $k \ge 1$. Let $G = (V, E)$ be a connected topological graph in $\mathbb{R}^n$, and for all $j \in J$ assume $\pi_j \in C^k([0, l_j]; \mathbb{R}^n)$. Then the union*

$$\Gamma := \bigcup_{j \in J} \bar{e}_j \subset \mathbb{R}^n$$

*is called* the (topological) $c^k$-network $\Gamma$ belonging to $G$.

Observe that a topological $c^k$-network $\Gamma$ is a compact topological subspace of $\mathbb{R}^n$. Moreover, as the edge parametrizations $\pi_j$, $j \in J$, are at least $C^1$, it is clear that the topology induced by the path metric $d$ on $\Gamma$ is equivalent to the subspace topology.

In the sequel let $\Gamma$ always be a topological $c^\infty$-network and $G = (V, E)$ its underlying topological graph, with $V = \{v_i, i \in I\}$ and $E = \{e_j, j \in J\}$.

## 4.4 Boundary value problems on topological networks

We want to study boundary value problems on $\Gamma$. For this purpose we first specify what we mean by the *boundary* $\partial\Gamma$ of $\Gamma$. In fact we single out a non-empty index subset $I_B \subset I$ and define $\partial\Gamma := \{v_i, i \in I_B\} \subseteq V$ to be the set of *boundary vertices*. In contrast, we set $I_T := I \backslash I_B$ and call $\{v_i, i \in I_T\}$ the set of *transition vertices*.

In terms of interaction problems, the value of solutions is prescribed at boundary vertices, whereas at transition vertices the relation between the different solutions on the incident edges is put into relation by a transition condition. With this in mind, transition conditions at vertices with only one incident edge do not make sense. Let us therefore agree to demand $i \in I_B$ for each $i \in I$ with $\deg(v_i) = 1$.

Now for any function $u : \Gamma \to \mathbb{R}$ and each $j \in J$ we denote by $u^j$ the restriction of $u$ to $e_j$, i.e.

$$u^j := u \circ \pi_j : [0, l_j] \to \mathbb{R}.$$

The $C^\infty$-regularity of the parametrizations $\pi_j$ of a $\Gamma$ allows to differentiate along the edges, where differentiation along $e_j$ will be denoted by $\partial_j$, $j \in J$, that is we define

$$\partial_j^\alpha u(x) = \partial_j^\alpha u^j(\pi_j^{-1}(x)) := \left(\frac{\partial}{\partial x}\right)^\alpha u^j(\pi_j^{-1}(x))$$

for all $x \in e_j$ and all $\alpha \in \mathbb{N}$. At a given vertex $v_i$, $i \in I$, we furthermore define

$$\partial_j u(v_i) = \partial_j u^j(\pi_j^{-1}(v_i)) := \frac{\partial}{\partial x} u^j(\pi_j^{-1}(v_i)).$$

We now introduce the function spaces on $\Gamma$ we are mainly going to work with.

**Definition 4.6** *Let $u : \Gamma \to \mathbb{R}$.*

*We call $u$* continuous*, if $u$ is continuous with respect to the subspace topology of $\Gamma$ induced by $\mathbb{R}^n$. We write $u \in C(\Gamma)$.*

*We call $u$ $k$* times differentiable*, $k \geq 1$, if $u^j \in C^k([0, l_j])$ for all $j \in J$. We write $u \in C^k(\Gamma)$.*

**Remark.** The sufficient condition that a collection of continuous functions $u^j : [0, l_j] \to \mathbb{R}$, $j \in J$, constitutes a function $u \in C(\Gamma)$ is given by

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \quad \text{whenever} \quad v_i \operatorname{inc} e_j \quad \text{and} \quad v_i \operatorname{inc} e_k.$$

In this sense, continuity can be regarded as a transition condition. In fact, it is the basic transition condition we require in the sequel.

As has already been announced, the *Kirchhoff condition* known from electrical circuits theory plays a fundamental role in the follwing considerations. In a way, $C^1$-differentiability along the edges means that left- and right-sided derivatives are related in a fashion that cusps are ruled out. In other words, the slopes in outward (or inward) direction with respect to each given point add up to zero. At vertices, this condition naturally generalizes to the Kirchhoff condition, the simplest form of a linear transition condition.

**Definition 4.7** *Let $u \in C^1(\Gamma)$, let $i \in I$ and $j \in \mathrm{Inc_i} := \{\mathrm{j} \in \mathrm{J} \mid \mathrm{v_i}\, inc\, \mathrm{e_j}\}$. We then set*

$$s_{ij}(u) := a_{ij}\partial_j u(v_i),$$

*where $(a_{ij})$ as defined in (4.1) is the incidence matrix of the topological graph corresponding to $\Gamma$.*

*Furthermore we define the linear mapping $S_i : C^1(\Gamma) \to \mathbb{R}$ by*

$$S_i(u) := \sum_{j \in \mathrm{Inc_i}} s_{ij}(u).$$

*We say that $u$ satisfies the* Kirchhoff condition *at $v_i$, $i \in I$, if $S_i(u) = 0$. We say that $u$ satisfies the Kirchhoff condition, if it is satisfied for all $i \in I_T$ and write $u \in C^1_K(\Gamma)$. For $k > 1$ we moreover set*

$$C^k_K(\Gamma) := C^k(\Gamma) \cap C^1_K(\Gamma).$$

## 4.5 Maximum and comparison principles for Kirchhoff functions

If we ask the Kirchhoff condition at transition vertices to be satisfied, several standard maximum principles for linear and semilinear equations can be carried over to networks. For later purposes we start with the following simple observation about Kirchhoff functions.

**Lemma 4.1** *Let $k \in \mathbb{N}$ and $k \geq 1$. Then the space $C^k_K(\Gamma)$ forms an algebra.*

**Proof.** Let $f, g \in C^k_K(\Gamma)$. Then we have for all $i \in I_T$

$$\begin{aligned} S_i(fg) &= \sum_{j \in \mathrm{Inc_i}} a_{ij}\partial_j(fg)(v_i) = \sum_{j \in \mathrm{Inc_i}} a_{ij} f(v_i)\partial_j g(v_i) + \sum_{j \in \mathrm{Inc_i}} a_{ij} g(v_i)\partial_j f(v_i) \\ &= f(v_i)S_i(g)(x) + g(v_i)S(f)(x) = 0. \end{aligned}$$

□

The following lemma states a maximum principle for certain linear operators on networks.

**Lemma 4.2** *Let $L := (L^j)_{j\in J}$ be a collection of linear differential operators given by*

$$L^j(f) := a^j\partial_j^2 f + b^j\partial_j f, \quad f \in C^2(]0, l_j[),$$

*with coefficient functions $a^j, b^j : ]0, l_j[ \to \mathbb{R}$, $j \in J$. Assume $L$ to be uniformly elliptic in the sense that there are constants $\lambda > 0$, $\Lambda > 0$ such that $\lambda \le a^j \le \Lambda$ on $]0, l_j[$ for all $j \in J$. Furthermore assume that there is a constant $C(\lambda)$ such that $|b^j| \le C(\lambda)$ on $]0, l_j[$ for all $j \in J$.*

*Let $u \in C^2(\Gamma)$ such that*

*(i) $L^j(u^j) \ge 0$ ($L^j(u^j) \le 0$) on $]0, l_j[$ for all $j \in J$ and*
*(ii) $S_i(u) \ge 0$ ($S_i(u) \le 0$) for all $i \in I_T$.*

*Then we have $\max_{\partial\Gamma} u = \max_\Gamma u$, ($\min_{\partial\Gamma} u = \min_\Gamma u$), where $\partial\Gamma := \{v_i \,|\, i \in I_B\}$.*

**Remark.** In particular, the lemma holds for all $u \in C^2_K(\Gamma)$ with $L^j(u^j) \le 0$ for all $j \in J$.

For the proof of lemma 4.2 we need the following proposition.

**Proposition 4.1** *Let $L$ be as in lemma 4.2. Then there is a function $f \in C^2(\Gamma)$ satisfying*

$$\begin{aligned} &L^j(f^j) > 0 \quad \text{on } ]0, l_j[ \text{ for all } j \in J,\\ &S_i(f) > 0 \quad \text{for all } i \in I_T. \end{aligned}$$

**Proof.** (of proposition 4.1). Let $\gamma > 0$ such that we have $\lambda\gamma^2 - C(\lambda)\gamma > 0$. Set $p := |I|$ and define the set

$$M := \{\xi \in \mathbb{R}^p \text{ with } \xi_i \ne \xi_j \text{ if } v_i \operatorname{adj} v_j, i, j \in I\}. \tag{4.2}$$

Let $\xi \in M$. Fix $k \in J$ and let $i, j \in I$ such that $e_k = v_i v_j$. Since $\xi_i \ne \xi_j$, we can find unique numbers $\sigma, \eta, c \in \mathbb{R}$ with $|\sigma| = 1$ such that the function

$$u^k : [0, l_k] \to \mathbb{R}, \quad u^k(x) := e^{\sigma\gamma(x-\eta)} + c$$

satisfies $u^k(\pi_k^{-1}(v_i)) = \xi_i$ and $u^k(\pi_k^{-1}(v_j)) = \xi_j$.

We then compute

$$L^k(u^k)(x) = (a^k(x)\lambda^2 + b^k(x)\sigma\lambda)e^{\sigma\lambda(x-\eta)} \quad \text{for all } x \in ]0, l_k[.$$

By the choice of $\gamma$ it then follows

$$L^k(u^k) > (\lambda\gamma^2 - C(\lambda)\gamma)e^{\sigma\lambda(x-\eta)} > 0$$

If we repeat this for all other choices $k \in J$, we obtain an injective mapping

$$\Phi : M \to D \quad \text{with} \quad D := \{u \in C^2(\Gamma) \text{ with } L^j(u^j) > 0 \text{ on } ]0, l_j[ \text{ for all } j \in J\},$$

satisyfing $\Phi(\xi)(v_i) = \xi_i$ for all $i \in I$ and all $\xi \in M$.

Now we show that we can choose $\xi = (\xi_i) \in M$ such that $S_i(\Phi(\xi)) > 0$ is satisfied for all $i \in I_T$. To this end observe that for $i \in I_T$ the mapping

$$T_i : M \to \mathbb{R}, \quad T_i := S_i \circ \Phi, \tag{4.3}$$

is a continuous, strictly decreasing and unbounded function in the component $\xi_i$. Furthermore observe that $T_i$ is continuous, unbounded, and strictly increasing in each component $\xi_j$, $j \in A_i$, where $A_i := \{j \in I \,|\, v_j \text{ adj } v_i\}$. Finally, $T_i$ is independent of the component $\xi_j$ for any $j \in I \backslash (\{i\} \cup A_i)$.

We construct $\xi \in M$ such that $T_i(\xi) > 0$ for all $i \in I_T$. Let dist : $I \times I \to \mathbb{N}$ be the metric given by the smallest number $\mathrm{dist}(i, j)$ of vertices a path connecting $v_i$ and $v_j$ has to visit. It induces the partition $I_l := \{i \in I \,|\, \mathrm{dist}(i, I_B) = l\}$, $l \in \mathbb{N}_0$. Observe that $I_0 = I_B$. Let $m := \max\{l \in \mathbb{N}_0 \,|\, I_l \neq \emptyset\}$. Furthermore note that for $i \in I_l$, $1 \leq l \leq m$, there is by construction at least one $j \in I_{l-1}$ such that $j \in A_i$. Moreover, $T_i$ is constant in $\xi_j$ for all $j \in I \backslash (I_{l-1} \cup I_l \cup I_{l+1})$.

We first choose $\xi_i$ arbitrarily for all $i \in I_m$. Due to the fact that $T_i$ is unbounded, continuous, and strictly increasing in each $\xi_j$, $j \in A_i$, and by the fact that $I_{m-1} \cap A_i \neq \emptyset$ for each $i \in I_m$, we then may choose the components $\xi_j$, $j \in I_{m-1}$, sufficiently large to have $T_i > 0$ for all $i \in I_m$. Now choose $\xi_j$, $j \in I_{m-2}$, large enough to ensure that $T_i > 0$ for all $i \in I_{m-1}$. For $k = 3, \ldots, m$ we continue this procedure by choosing $\xi_j$, $j \in I_{m-k}$, large enough to ensure that $T_i > 0$ for all $i \in I_{m-k+1}$, ending up with a choice for $\xi \in M$ such that $T_i(\xi) > 0$ for all $i \in \cup_{l=1}^m I_l = I_T$. Setting $f := \Phi(\xi)$ completes the proof. □

**Proof.** (of lemma 4.2). First assume $L^j(u^j) > 0$ on $]0, l_j[$ for all $j \in J$ and $S_i(u) > 0$ for all $i \in I_T$. Suppose that there be some $j \in J$ and some $x_0 \in ]0, l_j[$ such that $u^j$ attains a local maximum at $x_0$. It follows $\partial_j u^j(x_0) = 0$ and $a^j \partial_j^2 u^j(x_0) \leq 0$, which contradicts the assumption $L^j(u^j)(x_0) > 0$. Now assume that there be some $i \in I_T$ such that $u$ attains a local maximum at $v_i$. Then $a_{ij} \partial_j u^j(\pi_j^{-1}(v_i)) \leq 0$ for all $j \in \mathrm{Inc_j}$, whence we have $S_i(u)(v_i) \leq 0$, a contradiction to our assumption $S_i(u) > 0$ for all $i \in I_T$.

Now assume $L^j(u^j) \geq 0$ on $]0, l_j[$ for all $j \in J$ as well as $S_i(u) \geq 0$ for all $i \in I_T$. For $\delta > 0$ we then set $w_\delta := u + \delta f \in C^2(\Gamma)$, where $f$ is the function constructed in proposition 4.1. By linearity of $L^j$, $j \in J$, and $S_i$, $i \in I_T$, we have $L^j(w_\delta^j) > 0$ on $]0, l_j[$ for all $j \in J$ as well as $S_i(w_\delta) > 0$ for all $i \in I_T$. By the arguments above it follows that $\max_\Gamma(u+\delta f) = \max_{\partial\Gamma}(u+\delta f)$, whence by passing to the limit $\delta \to 0$ we obtain $\max_\Gamma u = \max_{\partial\Gamma} u$.

Finally, assume $L^j(u^j) \leq 0$ on $]0, l_j[$ for all $j \in J$ as well as $S_i(u) \leq 0$ for all $i \in I_T$. By linearity it follows $L^j(-u^j) \geq 0$ on $]0, l_j[$ for all $j \in J$ as well as $S_i(-u) \geq 0$ for all $i \in I_T$. We obtain $\min_\Gamma u = \max_\Gamma -u = \max_{\partial\Gamma} -u = \min_{\partial\Gamma} u$. □

From lemma 4.2 we now derive a comparison result for certain semilinear operators on networks.

**Lemma 4.3** *Let $Q = (Q_j)_{j\in J}$ be a collection of semilinear operators given by*

$$Q^j(f)(x) := a^j(x)\partial_j^2 f(x) + b^j(\partial_j f^j(x), f^j(x), x), \quad x \in ]0, l_j[, f \in C^2(\,]0, l_j[\,),$$

*with coefficient functions $a^j : ]0, l_j[\to \mathbb{R}$ and $b^j : \mathbb{R} \times \mathbb{R}\times ]0, l_j[\to \mathbb{R}$, $j \in J$. For all $j \in J$ assume furthermore that $b^j(\cdot, z, x) \in C^1(\mathbb{R})$ for all $(z, x) \in \mathbb{R}\times ]0, l_j[$, and that $b^j(p, \cdot, x)$ is non-increasing for all $(p, x) \in \mathbb{R}\times ]0, l_j[$.*

*Assume we are given two functions $u, v \in C^2(\Gamma)$ such that*

$$Q^j(u^j) \geq Q^j(v^j)$$

*for all $j \in J$ as well as $S_i(u) \geq S_i(v)$ for all $i \in I_T$. Furthermore suppose $u \leq v$ on $\partial\Gamma$. Then we have $u \leq v$ on $\Gamma$.*

**Proof.** For all $j \in J$ and all $x \in ]0, l_j[$ we have

$$\begin{aligned}
& Q^j(u^j)(x) - Q^j(v^j)(x) \\
= \; & a^j(x)\partial_j(u^j(x) - v^j(x)) + b^j(\partial_j u^j(x), u^j(x), x) - b^j(\partial_j v^j(x), v^j(x), x) \\
= \; & a^j(x)\partial_j(u^j(x) - v^j(x)) + b^j(\partial_j u^j(x), u^j(x), x) - b^j(\partial_j v^j(x), u^j(x), x) \\
& + b^j(\partial_j v^j(x), u^j(x), x) - b^j(\partial_j v^j(x), v^j(x), x) \geq 0, \qquad (4.4)
\end{aligned}$$

whenever $u^j(x) > v^j(x)$, as $b^j(p, \cdot, x)$ is non-increasing. Since $b^j(\cdot, z, x)$ is continuously differentiable, for each $j \in J$ there is a locally bounded function $\tilde{b}^j : ]0, l_j[\to \mathbb{R}$, such that

$$b^j(\partial_j u^j(x), u^j(x), x) - b^j(\partial_j v^j(x), u^j(x), x) = \tilde{b}^j \partial_j(u^j(x) - v^j(x))$$

by the mean value theorem. Defining $w \in C^2(\Gamma)$ by $w := u - v$, by (4.4) we obtain

$$L^j(w^j) := a^j \partial_j^2 w^j + \tilde{b}^j \partial_j w^k \geq 0 \quad \text{on } ]0, l_j[\cap A, \, j \in J,$$

where $A := \{w > 0\} \subseteq \Gamma$. By linearity of $S_i$ we furthermore have $S_i(w) \geq 0$ for all $i \in I_T$ and $w \leq 0$ on $\partial\Omega$. By the proof of lemma 4.2 it follows that $w$ cannot attain a local maximum on the (open) set $A$, and as we have $A \cap \partial\Gamma = \emptyset$, it follows $A = \emptyset$ and thus $u \leq v$ on $\Gamma$. □

## 4.6 The viscous eikonal equation on networks

Having introduced the framework of topological networks, let us now return to the track we have outlined above - the idea of applying the method of vanishing viscosity to nonlinear first order boundary value problems on topological networks. In the present section, before considering the general case, we treat the exemplary case of the viscous eikonal equation and show existence and uniqueness of solutions satisfying the Kirchhoff condition. The proofs are elementary, but serve as an illustration of the importance of the Kirchhoff condition in order to make solutions unique. Later it will turn out that the Kirchhoff condition also is a sufficient condition to ensure the convergence of the vanishing viscosity method.

Accordingly, we are interested in existence and uniqueness of solutions $u \in C^2_K(\Gamma)$ of the following boundary value problem on $\Gamma$.

$$\begin{aligned} \varepsilon\partial_j^2 u + 1 - (\partial_j u)^2 &= 0 \qquad \text{on } e_j \text{ for all } j \in J, \\ u(v_i) &= g_i \qquad \text{for all } i \in I_B, \quad \varepsilon > 0, \end{aligned} \tag{4.5}$$

where $g_i \in \mathbb{R}$, $i \in I_B$.

**Theorem 4.1** *There is a unique function $u \in C^2_K(\Gamma)$ solving problem 4.5.*

The proof is given by the following collection of results.

**Lemma 4.4** *Let $u, v \in C^2(\Gamma)$ be solutions of boundary value problem 4.5. Then $u \equiv v$ on $\Gamma$.*

**Proof.** Observe that $Q = (Q^j)_{j\in J}$ with

$$Q^j(u^j) := \varepsilon\partial_j^2 u^j + 1 - (\partial_j u^j)^2, \quad j \in J,$$

satisfies the conditions of lemma 4.3. Then the assertion is an immediate consequence of this lemma. □

We now show the existence of solutions of (4.5).

**Proposition 4.2** *Let $a, b \in \mathbb{R}$, $a < b$. Then there is an injective mapping*

$$\Psi : \mathbb{R} \times \mathbb{R} \to C^2([a, b]),$$

*such that for each pair $(s, t) \in \mathbb{R} \times \mathbb{R}$ we have $u(a) = s$, $u(b) = t$, and*

$$\varepsilon \frac{\partial^2}{\partial x^2} u - \left(\frac{\partial}{\partial x} u\right)^2 + 1 = 0 \quad \text{on } ]a, b[,$$

*where $u := \Psi(s, t)$.*

*Furthermore, define the functions*

$$\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \psi(s, t) := \frac{\partial}{\partial x}\Psi(s, t)(a)$$

*and*

$$\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \pi(s, t) := \frac{\partial}{\partial x}\Psi(s, t)(b).$$

*Then for all $s \in \mathbb{R}$ the functions $\psi(s, \cdot) : \mathbb{R} \to \mathbb{R}$ and $\pi(s, \cdot) : \mathbb{R} \to \mathbb{R}$ are continuous and strictly increasing. Furthermore for all $t \in \mathbb{R}$ the functions $\psi(\cdot, t) : \mathbb{R} \to \mathbb{R}$ and $\pi(\cdot, t) : \mathbb{R} \to \mathbb{R}$ are continuous and strictly decreasing.*

**Proposition 4.3** *There is a function $f \in C^2(\Gamma)$ such that there is a vector $(a_j)_{j\in J}$ with $a_j \neq 0$ for all $j \in J$ and such that*

$$\begin{aligned} \partial_j f &\equiv a_j && \text{on } e_j,\ j \in J, \\ S_i(f) &> 0 && \text{for each } i \in I_T. \end{aligned}$$

**Proof.** We proceed similar to the proof of proposition 4.1. Define the set $M$ as in (4.2) and observe that there is a canonical injective mapping

$$\Phi : M \to D := \{u \in C^2(\Gamma) \,|\, \exists\, (a_j)_{j\in J} \text{ such that } a_j \neq 0 \text{ and } \partial_j f \equiv a_j \text{ on } e_j,\ j \in J.\}$$

with $\Phi(\xi)(v_i) = \xi_i$, $i \in I$. It then suffices to show that there is a $\xi \in M$ such that $S_i(\Phi(\xi)) > 0$ for all $i \in I_T$. For this purpose define for all $i \in I_T$ the functions $T_i$ as in (4.3) and observe that they have the same properties as described in the proof of proposition 4.1. We then proceed exactly as in this proof to construct $\xi$. □

**Lemma 4.5** *There exists a solution $u \in C^2_K(\Gamma)$ for the boundary value problem 4.5.*

**Proof.** By proposition 4.2 there is an injective mapping

$$\Phi : \mathbb{R}^I \to C^2(\Gamma)$$

such that for each $\xi = (\xi_i)_{i\in I}$ and $u := \Phi(\xi)$ we have $u(v_i) = \xi_i$ for all $i \in I_T$ as well as

$$\varepsilon \partial_j^2 u + 1 - (\partial_j u)^2 = 0 \quad \text{on } e_j \text{ for all } j \in J.$$

Fix $i \in I_T$. Then the mapping

$$T_i : \mathbb{R}^I \to \mathbb{R}, \quad T_i := S_i \circ \Phi = \sum_{j\in \mathrm{Inc_i}} (s_{ij} \circ \Phi),$$

is continuous, as the mappings $s_{ij} \circ \Phi : \mathbb{R}^I \to \mathbb{R}$, $j \in \mathrm{Inc_i}$, (cf. definition 4.7) are continuous by proposition 4.2. It follows that the set

$$C := \{\xi \in \mathbb{R}^p \,|\, T_i(\xi) \geq 0,\ \forall\, i \in I_T,\ \xi_i \leq g_i,\ \forall\, i \in I_B\}$$

is closed. Furthermore $C$ is non-empty, as we clearly have $\xi^0 = (\xi_i^0)_{i\in I} \in C$ with $\xi_i^0 = \min_{k\in I_B} g_k$ for all $i \in I$. Finally, we show that

$$\sup_{\xi\in C} \max_{i\in I} |\xi_i| < \infty. \tag{4.6}$$

For this purpose let $f$ and $(a_j)_{j\in J}$ be the function and the corresponding vector as constructed in proposition 4.3 and let $a := \min_{j\in J} |a_j| > 0$. Furthermore set $g := \max_{i\in I_B} g_i$. For the function

$$\tilde{f} := -f/a + g - \min_{i\in I_B} f(v_i) \in C^2(\Gamma)$$

we then have

$$Q^j(f^j) := \varepsilon\partial_j^2 f^j + 1 - (\partial_j f^j)^2 = 1 - (a^j/a)^2 \leq 0$$

for all $j \in J$ as well as $S_i(\tilde{f}) \leq 0$ for all $i \in I_T$ by proposition 4.3. Now let $\xi \in C$ and $u := \Phi(\xi)$. By the properties of $u$ and by lemma 4.3 we then have $u \leq \tilde{f}$ on $\Gamma$. Hence $\xi_i \leq \tilde{f}(v_i)$ for all $i \in I$ and (4.6) is proved.

Define $\tilde{\xi} := (\tilde{\xi}_i)_{i\in I}$ by $\tilde{\xi}_i := \sup_{\xi\in C}\xi_i$. We have $\tilde{\xi} \in \mathbb{R}^I$, since $\tilde{\xi}_i < \infty$ for each $i \in I$ by (4.6). We show $\tilde{\xi} \in C$. As $C$ is closed, it suffices to show that there is a sequence $(\xi^n)_{n\in\mathbb{N}}$ in $C$ converging to $\tilde{\xi}$. For this purpose choose for each $i \in I$ a sequence $(\xi^{i;n})_{n\in\mathbb{N}}$ in $C$ such that $\lim_{n\in\mathbb{N}}\xi_i^{i;n} \to \tilde{\xi}_i$. Then for each $n \in \mathbb{N}$ define $\xi^n \in \mathbb{R}^I$ to be the componentwise maximum of the vectors $\xi^{i;n}$, $i \in I$. It follows $\lim_{n\to\infty}\xi^n = \tilde{\xi}$, whence it remains to show that $\xi^n \in C$ for all $n \in \mathbb{N}$. To this end we fix two vectors $\xi, \bar{\xi} \in C$ and verify that their componentwise maximum $\zeta$ is contained in $C$. Fix $i \in I_T$ and assume without restriction that $\zeta_i = \xi_i$. Observe that for all $j \in A_i := \{j \in I \mid v_j \operatorname{adj} v_i\}$ the function $s_{ij} \circ \Phi : \mathbb{R}^I \to \mathbb{R}$ is strictly increasing in the component $\xi_j$ by proposition 4.2. Hence the function $\xi \mapsto T_i(\xi)$, $\xi = (\xi_j)_{j\in I}$, is strictly increasing in each component $\xi_j$, $j \in A_i$. Therefore, as $\xi_i = \zeta_i$ and $\xi_j \leq \zeta_j$ for all $j \in A_i$, it follows $T_i(\zeta) \geq T_i(\xi) \geq 0$. As $i \in I_T$ has been chosen arbitrarily, we obtain $\zeta \in C$.

Now suppose that there is some $i \in I_T$ with $T_i(\tilde{\xi}) > 0$. By continuity of $T_i$ there is a $\xi \in \mathbb{R}^I$ with $\xi_j = \tilde{\xi}_j$ for all $j \in I\backslash\{i\}$ and $\xi_i > \tilde{\xi}_i$ such that $T_i(\xi) > 0$. Furthermore for all $j \in I_T \cap A_i$ we have $T_j(\xi) \geq 0$, since $T_j$ is strictly increasing in $\xi_i$. Moreover, $T_j(\tilde{\xi}) = T_j(\xi)$ for all $j \in I_T\backslash A_i$. It follows $\xi \in C$, a contradiction to the definition of $\tilde{\xi}$.

One derives a similar contradiction in the case that there is an $i \in I_B$ with $\tilde{\xi}_i < g_i$. Thus it follows $T_i(\tilde{\xi}) = 0$ for all $i \in I_T$ as well as $\tilde{\xi}_i = g_i$ for all $i \in I_B$. Consequently, $u := \Phi(\tilde{\xi})$ solves the boundary value problem 4.5. □

## 4.7 Convergence of the vanishing viscosity method on networks

Denoting the unique solution of boundary value problem 4.5 by $u_\varepsilon$, we now examine whether and in which sense the functions $u_\varepsilon$ converge to a limit function as $\varepsilon \to 0$, and which properties this limit function will possess. In the special case $g_i = 0$ for all $i \in I_B$, it will turn out that the $u_\varepsilon$ will converge to the distance function $d$ on $\Gamma$, as expected.

In the present section we will extend our point of view from the special case of the eikonal equation towards a general approach to first order Hamilton-Jacobi equations *of eikonal type*. To be precise, we study the convergence of solutions $u_\varepsilon$ of the following boundary value problem as $\varepsilon \to 0$, where the functions $u_\varepsilon$ are assumed to satisfy the Kirchhoff condition at the transition vertices.

$$\begin{aligned} \varepsilon\partial_j^2 u_\varepsilon^j(x) - H^j(\partial_j u_\varepsilon^j(x), u_\varepsilon^j(x), x) &= 0 \qquad \text{for all } x \in ]0, l_j[ \text{ and } j \in J \\ u_\varepsilon(v_i) &= g_i \qquad \text{for all } i \in I_B, \quad \varepsilon > 0. \end{aligned} \tag{4.7}$$

### 4.7.1 Convergence of $u_\varepsilon$

In the present section we establish our convergence result for the vanishing viscosity method on networks, that is, the convergence of solutions $u_\varepsilon$ of the boundary value problem 4.7. To this end we first prescribe a collection of prerequisites for the Hamiltonian $H^j$, $j \in J$, which we will assume to be valid from now on.

In fact we assume the functions $H^j : \mathbb{R} \times \mathbb{R}\times ]0, l_j[ \to \mathbb{R}$ to have the following properties for all $j \in J$:

$$\begin{array}{ll} (i) & H^j(0,z,x) < 0 \text{ for all } (z,x) \in \mathbb{R}\times ]0,l_j[ \\ (ii) & H^j \in C^2(\mathbb{R} \times \mathbb{R}\times ]0,l_j[ \\ (iii) & H^j(p,\cdot,x) \text{ is non-decreasing for all fixed choices of } (p,x) \in \mathbb{R}\times ]0,l_j[ \\ (iv) & H^j(p,z,x) \to \infty \text{ as } |p| \to \infty \text{ for all } (z,x) \in \mathbb{R}\times ]0,l_j[ \end{array} \tag{4.8}$$

We furthermore make the following demand on $H$ at transition vertices.

$$(v) \quad H^j(p,z,\pi_j^{-1}(v_i)) = H^j(-p,z,\pi_j^{-1}(v_i)) \text{ for all } i \in I_T,\ j \in \mathrm{Inc_i}. \tag{4.9}$$

**Remark.** The role of each of the conditions (i) - (v) will become clear in the subsequent proof. Later on we will provide some examples where some of these conditions are violated and the $u_\varepsilon$ do not converge.

As has been mentioned above, the limit function will not necessarily attain the boundary values $g_i$. We therefore give the following sufficient condition on $g_i$, $i \in I_B$, such that the assumption of the boundary values is ensured.

There is a constant $\delta > 0$ and a function $\psi : \Gamma \to \mathbb{R}$ with $\psi^j \in C^2([0, l_j])$ for all $j \in J$ such that we have

$$\begin{aligned} &\psi(v_i) = g_i \text{ for all } i \in I_B, \\ &H^j(\partial_j\psi^j, \psi^j, x) < -\delta \quad \text{on } ]0,l_j[ \text{ for all } j \in J, \\ &S_i(\psi) \geq 0 \text{ for all } i \in I_T. \end{aligned} \tag{4.10}$$

**Remark.** As this condition is somewhat abstract, we will discuss alternative conditions on the boundary values $g_i$ and the topology of the graph $\Gamma$ which imply condition 4.10 and are easier to describe.

**Lemma 4.6** *Assume that for each $\varepsilon > 0$ we have a solution $u_\varepsilon \in C_K^2(\Gamma)$ for the boundary value problem 4.7 and assume condition 4.10 to be satisfied. Then there is an $\tilde{\varepsilon} > 0$ such that the functions $u_\varepsilon, 0 < \varepsilon < \tilde{\varepsilon}$, are uniformly bounded in $\varepsilon$ and equicontinuous on $\Gamma$.*

**Theorem 4.2** *Under the conditions of lemma 4.6 the functions $u_\varepsilon$ uniformly converge to a limit function $u \in C(\Gamma)$ as $\varepsilon \to 0$.*

**Proof.** (Theorem 4.2) First recall that $\Gamma$ endowed with the induced topology of $\mathbb{R}^n$ is a compact space. By lemma 4.6 the sets $B(x) := \{u_\varepsilon(x) \,|\, 0 < \varepsilon < \tilde{\varepsilon}\}$, $x \in \Gamma$, are relatively compact. This and the equicontinuity of $u_\varepsilon$ imply the assertion by the theorem of Arzela-Ascoli. □

**Proof.** (Lemma 4.6) We proceed in several steps.

*1. Bounding $|u_\varepsilon|$ uniformly in $\varepsilon$*

The idea is to construct functions $\bar{v}$ and $\underline{v}$ constituting uniform upper and lower bounds, respectively, for the family $(u_\varepsilon)$, $0 < \varepsilon < \tilde{\varepsilon}$. In fact, we choose $\underline{v} :\equiv m := \min_{i \in I_B} g_i$. Then it follows for all $j \in J$ and $\varepsilon > 0$

$$Q^j_\varepsilon(\underline{v}) := \varepsilon \partial_j^2 \psi^j - H^j(\partial_j \psi^j, \psi^j, x) = -H^j(0, m, x) > 0 = Q_\varepsilon(u_\varepsilon),$$

where the last inequality is due to property 4.8 (i). Moreover, we have $S_i(\psi) = 0 = S_i(u_\varepsilon)$ for all $i \in I_T$ as well as $\psi(v_i) \leq g_i = u_\varepsilon(v_i)$ for all $i \in I_B$. Hence the comparison lemma 4.3 implies $u_\varepsilon \geq m$ on $\Gamma$.

In order to construct the function $\bar{v}$ we proceed similarly to the proof of proposition 4.1 such that $\bar{v}$ will end up having the following properties:

$$\begin{aligned} &\bar{v}^j \in C^2(]0, l_j[), \quad \forall j \in J, \\ &|\partial_j \bar{v}^j| \geq M, \quad \forall j \in J, \\ &S_i(\bar{v}) < 0, \\ &\bar{v}(v_i) \geq g_i \quad \forall i \in I_B, \end{aligned} \tag{4.11}$$

where $M$ is a positive number such that $H^j(p, m, x) > 0$ for all $j \in J$, $x \in [0, l_j]$, and $|p| = M$ ($m := \min_{i \in I_B} g_i$; the existence of $M$ is ensured by conditions 4.8).

For this purpose we again consider the partition $I_l := \{i \in I \,|\, \mathrm{dist}(i, I_B) = l\}$, $l \in \mathbb{N}_0$, where $\mathrm{dist} : I \times I \to \mathbb{N}$ be metric given by the smallest number $\mathrm{dist}(i, j)$ of vertices a path connecting $v_i$ and $v_j$ has to visit. Let $m := \max\{l \in \mathbb{N}_0 \,|\, I_l \neq \emptyset\}$. For all $i \in I_m$ we now choose values $w(v_i) \in \mathbb{R}$ arbitrarily. Due to the fact that we have $I_{m-1} \cap A_i \neq \emptyset$, where $A_i := \{j \in I \,|\, v_j \,\mathrm{adj} v_i\}$, we then may choose the values $w(v_i)$, $j \in I_{m-1}$, in such a way that we have

$$T_i := \sum_{j \in A_i} (w(v_j) - w(v_i))/d(v_j, v_i) < 0$$

for all $i \in I_m$. Similarly, we may choose values $w(v_j)$, $j \in I_{m-2}$, in such a way that $T_i < 0$ for all $i \in I_{m-1}$. Continuing this procedure ends up in a choice for $w(v_i)$ for all $i \in I$ such that $T_i < 0$ for all $i \in \cup_{l=1}^m I_l = I_T$. We also assume that all values $w(v_i)$, $i \in I$, are pairwise different, by possibly slightly perturbing them without violating the property $T_i < 0$. This is possible as the $T_i$ depend continuously on the $w(v_i)$. We now linearly extend the function $w$ defined on the vertices to the edges by setting for all $j \in J$

$$w^j(x) := w(v_i) + (w(v_k) - w(v_i))/d(v_i, v_k),$$

where $e_j = v_i v_k$, and we clearly have $S_i(w) = T_i < 0$ for all $i \in I_T$. Let $c > 0$ be the minimal modulus of the slope of $w$ on the edges. The function

$$\bar{v} := M/c \cdot w + C$$

then satisfies all requirements of 4.11, where $C > 0$ is to be chosen such that $\bar{v}(v_i) \geq g_i$ for all $i \in I_B$.

As a direct consequence of 4.11 and 4.8 we then have

$$Q^j_\varepsilon(\bar{v}^j) = -H^j(\partial_j \bar{v}^j, \bar{v}^j, x) \leq -H^j(p, m, x) < 0, \quad \text{where } |p| = M.$$

Furthermore we have $S_i(\bar{v}) < 0$ for all $i \in I_T$ and $\bar{v}(v_i) \geq g_i$ for all $i \in I_B$. The comparison lemma 4.3 then yields $u_\varepsilon \leq \max_\Gamma \bar{v}$ for all $\varepsilon > 0$.

Altogether we have

$$|u_\varepsilon| < C_1 \tag{4.12}$$

for all $\varepsilon > 0$ and a constant $C_1 < \infty$.

*2. Bounding the first derivatives of $u_\varepsilon$ uniformly in $\varepsilon$ at the boundary vertices*

We first derive a uniform bound for $|\partial_j u^j_\varepsilon(\pi_j^{-1}(v_i))|$ for all $i \in I_B$ and $j \in \mathrm{Inc_i}$. For this purpose we pinch the functions $u_\varepsilon$ between two fixed functions with bounded first derivatives at the boundary vertices. In fact, we show that there are constants $\kappa > 0$ and $\beta > 0$ such that

$$\psi \leq u_\varepsilon \leq \psi + \kappa\delta_0 \quad \text{on } \Gamma_\beta \text{ and for all } 0 < \varepsilon < \tilde{\varepsilon}, \tag{4.13}$$

where $\Gamma_\beta := \{x \in \Gamma \,|\, d(x, \partial\Gamma) \leq \beta\}$ and $\partial\Gamma := \{v_i \,|\, i \in I_B\}$. Furthermore, $\psi$ is the function the existence of which has been assumed in 4.10 and $\delta_0$ is the distance function. As $\psi^j \in C^2[0, l_j]$ for all $j \in J$, it follows that there is a constant $\tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon}\partial_j^2\psi^j > -\delta$, where $\delta$ is the constant defined in 4.10. Invoking 4.10 once more we then derive

$$Q_\varepsilon(\psi^j) = \varepsilon\partial_j^2 - H^j(\partial_j\psi^j, \psi^j, x) > 0 = Q_\varepsilon(u^j_\varepsilon)$$

for all $0 < \varepsilon \leq \tilde{\varepsilon}$, $j \in J$. Furthermore we have $S_i(\psi) \geq 0 = S_i(u_\varepsilon)$ for all $i \in I_T$ as well as $\psi(v_i) = g_i = u_\varepsilon(v_i)$ for all $i \in I_B$, and the comparison lemma 4.3 yields $u_\varepsilon \geq \psi$ on $\Gamma$ for all $0 < \varepsilon \leq \tilde{\varepsilon}$ and we have established the first part of inequality 4.13.

In order to derive the second part of 4.13 first observe that there is a constant $\beta > 0$ such that the distance function $\delta_0$ does not attain a local maximum on $\Gamma_\beta$. Fix arbitrary indices $i \in I_B$ and

$j \in \mathrm{Inc_i}$. Assuming without loss of generality that the edge $e_j$ be parametrized with $\pi_j(0) = v_i$, it follows that $|\partial_j \delta_0^j| \equiv 1$ and $\partial_j^2 \delta_0^j \equiv 0$ on $[0, \beta]$. For $\tilde{\psi} := \psi + \kappa\delta_0$ it therefore follows

$$Q_\varepsilon(\tilde{\psi} j) = \varepsilon \partial_j \psi^j - H^j(\partial_j \tilde{\psi}^j, \tilde{\psi}^j, x) \leq \tilde{\varepsilon} \max_{j \in J} \max_{x \in [0,l_j]} -H^j(\partial_j \tilde{\psi}^j, \tilde{\psi}^j, x) < 0 = Q_\varepsilon(u_\varepsilon^j), \tag{4.14}$$

for all $x \in [0, \beta]$, all $0 < \varepsilon \leq \tilde{\varepsilon}$, provided that $\kappa > 0$ is large enough. Here we have taken advantage of the properties of $H$ given by 4.8 (iii) and (iv). By possibly enlarging $\kappa$ we can also arrange that we have

$$\tilde{\psi}^j(\beta) \geq C_1, \tag{4.15}$$

where $C_1$ is the uniform bound of the $|u_\varepsilon|$ derived above. Then again, the comparison lemma 4.3 implies

$$\tilde{\psi}^j \geq u_\varepsilon^j \quad \text{on } [0, \beta]. \tag{4.16}$$

Of course we can choose $\kappa$ large enough such that 4.14 - 4.16 hold for all $i \in I_B$ and $j \in \mathrm{Inc_i}$, which implies the second part of relation 4.13. Consequently we have

$$|\partial_j u_\varepsilon(v_i)| < C_2 \tag{4.17}$$

for all $i \in I_B$, for all $0 < \varepsilon < \tilde{\varepsilon}$ and a constant $C_2 < \infty$.

*3. Bounding the first derivatives of $u_\varepsilon$ at the transition vertices*

At each transition vertex $i \in I_T$ we now derive a uniform bound of $|\partial_j u_\varepsilon^j|$ for all $j \in \mathrm{Inc_i}$ and all small $\varepsilon > 0$. For this purpose we derive a contradiction by assuming that for a fixed index $i \in I_T$ such a bound do not exist. Under this assumption, there is an index $k \in \mathrm{Inc_i}$ along with a sequence $\varepsilon_n \to 0$ such that $\lim_{n\to\infty} |\partial_k u_{\varepsilon_n}^k(\pi_k^{-1}(v_i))| = \infty$. As for each $n \in \mathbb{N}$ the Kirchhoff condition

$$S_i(u_{\varepsilon_n}) = \sum_{j \in \mathrm{Inc_i}} a_{ij} \partial_j u_{\varepsilon_n}^j(\pi_j^{-1}(v_i)) = 0$$

is satisfied, it follows that there is an index $j \in \mathrm{Inc_i}$ such that $\lim_{n\to\infty} a_{ij}\partial_j u_{\varepsilon_n}^j(\pi_j^{-1}(v_i)) = \infty$. Consequently there is a sequence $x_n \in e_j$ with $x_n \to v_i$ such that

$$\lim_{n\to\infty} a_{ij} \partial_j u_{\varepsilon_n}^j(y_n) = \infty, \tag{4.18}$$

where $y_n := \pi_j^{-1}(x_n)$. Observe that we can assume without loss of generality that there is a number $t_0 > 0$ such that the functions

$$f_n(t) := u_{\varepsilon_n}^j(y_n + a_{ij}t), \quad n \in \mathbb{N},$$

are well-defined for all $t \in [0, t_0]$ with $f_n \in C^2([0, t_0])$. Then 4.18 reads

$$\lim_{n\to\infty} f_n'(0) = \infty. \tag{4.19}$$

By 4.7 we have for all $n \in \mathbb{N}$

$$\begin{aligned} & \varepsilon_n f_n''(t) - H^j(a_{ij} f_n'(t), f_n(t), y_n + t) \\ = \quad & \varepsilon_n \partial_j^2 u_{\varepsilon_n}^j(y_n + a_{ij}t) - H^j(\partial_j u_{\varepsilon_n}^j(y_n + a_{ij}t), u_{\varepsilon_n}^j(y_n + a_{ij}t), y_n + a_{ij}t) = 0 \end{aligned}$$

or equivalently

$$f_n''(t) = \varepsilon_n^{-1} H^j(a_{ij} f_n'(t), f_n(t), y_n + t)$$

on $[0, t_0]$. Now observe that by 4.8 (iii) and 4.12 we have

$$f_n''(t) = \varepsilon_n^{-1} H^j(a_{ij} f_n'(t), f_n(t), y_n + t) \geq \varepsilon_n^{-1} \min_{y \in [0, l_j]} H^j(a_{ij} f_n'(t), -C_1, y). \tag{4.20}$$

Now fix $a > 0$. Then 4.8 (iv) implies that for all sufficiently large numbers $l_0 > 0$ we have

$$\min_{y \in [0, l_j]} H^j(l, -C_1, y) > a \tag{4.21}$$

for all $l \geq l_0$ and all $l \leq -l_0$. Hence, by 4.19, there is an $n \in \mathbb{N}$ such that 4.21 holds with $l_0 = f_n'(0) > 0$. Then 4.20 and 4.21 imply

$$f_n''(0) > a, \tag{4.22}$$

provided that $\varepsilon_n \leq 1$, which we from now on assume. We claim that we also have

$$f_n''(t) \geq a \quad \text{for all } t \in ]0, t_0]. \tag{4.23}$$

For if, assuming the contrary, there is a $\tilde{t} \in ]0, t_0[$ such that $f_n''(\tilde{t}) < a$, by 4.22 there must be a maximal $t$ with $0 < t < \tilde{t}$ and $f_n''(t) = a$ as well as $f_n''(s) < a$ for all $t < s \leq \tilde{t}$. Observe, however, that due to $f_n''(t) = a > 0$ there is a small number $\xi > 0$ such that $f_n'(t+s) > f_n'(t) > 0$ for all $s \in ]0, \xi[$. It then follows

$$f_n''(t+s) = \varepsilon_n^{-1} H^j(a_{ij} f_n'(t+s), f_n(t+s), y_n + t + s) \geq \varepsilon_n^{-1} \min_{y \in [0, l_j]} H^j(a_{ij} f_n'(t+s), -C_1, y) > a$$

by 4.21, a contradiction to the choice of $t$.

¿From 4.22 and 4.23 it follows that the inequality

$$f_n(t) \geq f_n''(0) t^2 + f_n'(0) t + f_n(0)$$

holds on $[0, t_0]$. This and the relations $f_n'(0) \geq 0$ and $f_n''(0) \geq a$ yield

$$u_{\varepsilon_n}^j(y_n + a_{ij} t_0) = f_n(t_0) > f_n(0) + a t_0^2 > -C_1 + a t_0^2,$$

where $C_1$ is the uniform bound of $|u_\varepsilon|$ on $\Gamma$ defined in 4.12. As the choice of $a > 0$ has been arbitrary, we now choose $a := 2C_1/t_0^2$ and obtain

$$u^j_{\varepsilon_n}(y_n + a_{ij}t_0) > C_1,$$

a contradiction to 4.12.

Consequently we have

$$|\partial_j u_\varepsilon(v_i)| < C_3 \tag{4.24}$$

for all $i \in I_T$, for all $0 < \varepsilon < \tilde{\varepsilon}$ and a constant $C_3 < \infty$.

*4. Bounding $|\partial_j u^j_\varepsilon|$, $j \in J$, uniformly in $\varepsilon$*

We finally derive a bound for $|\partial_j u^j_\varepsilon|$, $j \in J$, uniformly in $\varepsilon$. For this purpose we fix $j \in J$, set $w^j := \partial_j u^j_\varepsilon$, and differentiate equation 4.7 to obtain

$$\varepsilon \partial_j^2 w^j - H^j_p(\partial_j u^j_\varepsilon, u^j_\varepsilon, x)\partial_j w^j - H^j_z(\partial_j u^j_\varepsilon, u^j_\varepsilon, x)w^j - H^j_x(\partial_j u^j_\varepsilon, u^j_\varepsilon, x) = 0, \tag{4.25}$$

which is possible due to 4.8 (ii). Observe moreover that we have $H^j_z(\partial_j u^j_\varepsilon, u^j_\varepsilon, x) \geq 0$ due to 4.8 (iii). Let us additionally assume here that $H^j$ be constant in $x$. Then 4.25 simplifies to

$$aw^j + b\partial_j w^j + cw^j = 0$$

with bounded coefficient functions $a, b, c \in C(\mathbb{R} \times \mathbb{R} \times ]0, l_j[)$ and $c \leq 0$. Consequently, 4.17 and 4.24 and the classical maximum principle (cf. [9], corollary 3.2) imply

$$|\partial_j u_\varepsilon| < C_4 \quad \text{on } [0, l_j]$$

for all $0 < \varepsilon < \tilde{\varepsilon}$ and a constant $C_4 < \infty$. This completes the proof. □

### 4.7.2 Properties of the limit function

Apart from it being continuous, we do not know anything yet about the limit function $u$ the existence of which has been proved in theorem 4.2. In particular, we are interested in the questions if and in what sense $u$ satisfies

$$\begin{array}{ll} H^j(\partial_j u^j, u^j, x) = 0 & \text{on } ]0, l_j[,\ j \in J \\ u(v_i) = g_i, & i \in I_T, \end{array} \tag{4.26}$$

and which conditions are satisfied at the transition vertices.

Let us point out that the following chapter will present a self-contained theory based on test functions, which will answer these questions. However, in view of the historical development, let us at this stage adapt some arguments from Kružkov [5], which do not make use of the test function approach. In particular, we adapt the notion of Kružkov's generalized solutions to the situation of networks. We remark that in cases when proofs in this section are similar to the ones given in [5], we decided to restrict ourselves to merely outlining the arguments, whereas we go into greater detail as soon as crucial differences from Kružkov demand this.

Once again, let us for illustrational reasons start with the case of the eikonal equation. In fact, we have

**Lemma 4.7** *Let $H^j(\partial_j u^j, u^j, x) := |\partial_j u^j|^2 - 1$, $j \in J$, and let $g_i := 0$ for all $i \in I_B$. Then the limit function $u$ of theorem 4.2 coincides with the distance function $\delta_0$.*

**Proof.** We first show that $u$ cannot attain a strict local minimum on any (open) edge $e_j$, $j \in J$, or at a transition vertex $v_i$, $i \in I_T$. For this purpose assume that a strict local minimum is attained at a point $x \in \Gamma_0 := \Gamma \backslash \{v_i, i \in I_B\}$. Consequently there is a (open) neighborhood $B$ of $x$ such that $u(x) < u(y)$ for all $y \in B \backslash \{x\}$. In particular, there is a number $\delta > 0$ such that we have $u(x) + \delta < u(y)$ for all $y \in \partial B$. As the $u_\varepsilon$ uniformly converge to $u$ on $\Gamma$, it follows that we have $|u_\varepsilon(y) - u(y)| < \delta/4$ for all $y \in \bar{B}$ and all sufficiently small $\varepsilon > 0$. In particular we have

$$u_\varepsilon(y) > u(y) - \delta/4 > u(x) + \delta/4 > u_\varepsilon(x)$$

for all $y \in \partial B$. Consequently, $u_\varepsilon$ attains a local minimum in $B$ for sufficiently small $\varepsilon > 0$. This, however, contradicts to the fact that $u_\varepsilon$ cannot attain a local minimum on $\Gamma_0$, which follows from lemma 4.8. The assertion then follows from lemmas 4.9 and 4.10. □

**Lemma 4.8** *Let $u_\varepsilon$ be the solution of boundary value problem 4.7 for an arbitrary choice of $g_i$. Furthermore let $H$ satisfy the condition 4.8. Then $u_\varepsilon$ does not attain a local minimum on $\Gamma_0$.*

**Proof.** Suppose $u_\varepsilon^j$ attain a local minimum at $x \in ]0, l_j[$. Then we have $\partial_j^2 u_\varepsilon^j(x) \geq 0$ and $partial u_\varepsilon^j(x) = 0$. It follows

$$0 = \varepsilon \partial_j^2 u_\varepsilon^j(x) - H(\partial u_\varepsilon^j(x), u_\varepsilon^j(x), x) \geq -H(0, u_\varepsilon^j(x), x) > 0$$

by 4.8 (i), a contradiction.

Now suppose that $u_\varepsilon$ attain a local minimum at $v_i$, $i \in I_T$ and observe that by 4.8 (i) we may choose a number $\delta > 0$ such that $H^j(0, m, \pi_j^{-1}) < -\delta$ for all $j \in \mathrm{Inc_i}$ and $m := \min_\Gamma u_\varepsilon$. By 4.7 it follows $\partial u_\varepsilon^j(\pi_j^{-1}(x)) = 0$ for all $j \in \mathrm{Inc_i}$. For any $y \in ]0, l_j[$ with $d(y, v_i)$ sufficiently small we therefore have $H^j(\partial u_\varepsilon^j(y), u_\varepsilon^j(y), y) < -\delta/2$ due to continuity reasons. Moreover, it is clear that $\varepsilon \partial_j^2 u_\varepsilon^j(y)$ gets strictly larger than $-\delta/2$, if $d(y, v_i)$ is sufficiently small. Altogether it follows

$$0 = \varepsilon \partial_j^2 u_\varepsilon^j(y) - H(\partial u_\varepsilon^j(y), u_\varepsilon^j(y), y) > -\delta/2 + \delta/2 = 0,$$

a contradiction. □

**Lemma 4.9** *Let*

$$v \in X := \{v \in C(\Gamma) \,|\, v^j \in Lip([0,l_j]) \text{ for all } j \in J \text{ and } |\partial_j v^j| = 1 \text{ a. e. on } [0,l_j]\}$$

*and assume that* $v$ *do not attain a local minimum on* $\Gamma_0$*. Then* $v = \delta_0$*.*

**Proof.**

□

**Lemma 4.10** *Let* $H$ *satisfy 4.8. Then the limit function* $u$ *of theorem 4.2 solves 4.26 almost everywhere and takes on the boundary values* $g_i$*.*

**Proof.** □

### 4.7.3 Conditions on the boundary data

# Chapter 5

# Viscosity solutions on networks

**Summary.** In this chapter we present our theory of viscosity solutions of first order Hamilton-Jacobi equations of eikonal type on networks and derive existence and uniqueness results. Moreover we show that our notion of viscosity solutions is consistent with the method of vanishing viscosity.

## 5.1 Introduction

As we have seen in the previous chapter, a possible approach to select a solution for first order Hamilton-Jacobi euquations on networks (or domains) is the method of vanishing viscosity. However, two problems arise in this context: the problem of convergence and the problem of unicity. Whereas – under certain prerequisites – the convergence problem has been successfully solved in the previous chapter, the question of uniqueness is not easy to handle, especially in view of the arbitrary choice of the underlying network. For an adaptation of the uniqueness approach given by Kružkov in [5] has certain difficulties involved which depend on the topology of the network. In particular, as soon as cycles occur in the graph corresponding to the network, a direct adaptation of Kružkov's method fails or at least becomes unelegant.

On the other hand, an adaptation the theory of viscosity solutions based on test functions turns out to be both elegant and even more powerful. The major task in this context is to derive the correct transition conditions for solution candidates at the transition vertices, under providence of which we will only be able to fully adapt the viscosity solution concept. These transition conditions make up the core of our theory, as they – naturally – constitute the major difference of our approach from the existing theory. Let us elaborate here on an interesting phenomenon in this respect: The consistency of the viscosity solution concept with the method of vanishing viscosity will in fact be shown, which in other words means that any limit function $u$ of a converging sequence $u_\varepsilon$ in the spirit of theorem 4.2 will in fact be a viscosity solution. As the $u_\varepsilon$ satisfy the *Kirchhoff condition* at transition vertices, it at first might suggest itself to expect the limit function $u$ to do the same. However, this is by far not the case: whereas the Kirchhoff condition has a certain *averaging* effect on the functions $u_\varepsilon$, the correct transition condition for the limit function $u$ will thoroughly lose this averaging property in favour of another principle which only involves two incident branches at a time instead of all of them. Although at a first glance this might seem unexpected, it is coherent with the idea of vanishing viscosity which might equally be interpreted as "vanishing averages". In order to give an impression of the transition condition of the limit function, let us once again consider the eikonal equation on a network with zero boundary conditions. Of course the distance function $\delta_0$ to the boundary seems to be an appealing candidate for a solution. However, it generally does not satisfy a Kirchhoff condition at the vertices. It in fact displays a behaviour at the vertices which can be described to be governed by "finding the shortest way to the boundary" rather than "building local averages", a characterization which will also be applicable to the general case. In fact, imagine an additional edge to be inserted at a given transition vertex $v_i$. Then the distance function at $v_i$ and at the other incident edges will only be affected, if the new edge provides a shorter connection to a boundary vertex. Again it is remarkable how much this behaviour deviates from the Kirchhoff condition.

Any generalization of existing concepts to new situations has of course to be justified by checking if the very features which make up the success of the existing theory are preserved in the new setting. In the case of the theory of viscosity solutions, these features are unicity, existence, and consistency with the existing concepts, method of vanishing viscosity and Kružkov's generalized solutions. Giving this justification will be the main issue of the present chapter. In fact, our generalization of viscosity solution to networks will be just as weak to yield existence, while being sufficiently "selective" in order to ensure uniqueness. We will also justify that our viscosity solution is not only a technical construction but arises as a natural selection principle, which in particular selects the distance function on ramified spaces to be the unique viscosity solution of the above eikonal equation on graphs.

## 5.2 Hamilton-Jacobi equations on networks

Throughout this chapter let $\Gamma$ be a topological network with boundary index set $I_B \neq \emptyset$. For reasons of clarity let us once again state the boundary value problem we are concerned with. Our objective is to establish an appropriate weak theory of solutions of non-linear boundary value problems on $\Gamma$ of the form

$$\begin{aligned} &H^j(\partial_j u^j, u^j, x) = 0 && \text{on } ]0, l_j[,\ j \in J, \\ &u(v_i) = g_i, && i \in I_B, \end{aligned} \tag{5.1}$$

By now, continuity is the only property of a possible solution candidate for 5.1 which is reasonable to demand. All other properties will be established in the sequel.

Again we assume that the Hamiltonians $H^j$, $j \in J$, satisfy the following conditions:

$$
\begin{array}{ll}
(i) & H^j(0,z,x) < 0 \text{ for all } (z,x) \in \mathbb{R} \times [0,l_j] \\
(ii) & H^j \in C^2(\mathbb{R} \times \mathbb{R} \times [0,l_j]) \\
(iii) & H^j(p,\cdot,x) \text{ is non-decreasing for all } (p,x) \in \mathbb{R} \times [0,l_j] \\
(iv) & H^j(p,z,x) \to \infty \text{ as } |p| \to \infty \text{ for all } (z,x) \in \mathbb{R} \times [0,l_j] \\
(v) & H^j(p,z,x) \text{ is convex in } p \text{ for all fixed } (z,x) \in \mathbb{R} \times [0,l_j].
\end{array}
\tag{5.2}
$$

Moreover, we additionally assume the following conditions to hold at transition vertices.

$$
\begin{array}{ll}
(vi) & H^j(p,z,\pi_j^{-1}(v_i)) = H^j(-p,z,\pi_j^{-1}(v_i)) \text{ for all } i \in I_T,\ j \in \mathrm{Inc_i},\ (p,z) \in \mathbb{R} \times \mathbb{R} \\
(vii) & H^j(p,z,\pi_j^{-1}(v_i)) = H^k(p,z,\pi_k^{-1}(v_i)) \text{ for all } i \in I_T,\ j,k \in \mathrm{Inc_i},\ (p,z) \in \mathbb{R} \times \mathbb{R}.
\end{array}
\tag{5.3}
$$

**Remark.** Whereas condition (vii) is a natural continuity condition, the symmetry condition (vi) seems to be somewhat strong but cannot be relaxed as we will see later on. It is necessary to cope with the case when different edges incident to the same vertex have different orientations.

**Example.** Consider the eikonal equation with zero boundary data on $\Gamma$ given by

$$
\begin{array}{ll}
(\partial_j u^j)^2 - 1 = 0 & \text{on } ]0,l_j[,\ j \in J, \\
u(v_i) = 0 & \text{for all } i \in I_B.
\end{array}
$$

Here we have $H^j(p,z,x) := p^2 - 1$, and the $H^j$ clearly satisfy 5.2.

## 5.3 Preliminaries and Definitions

Before we give our definition of viscosity solutions, let us introduce some useful terminology capturing and simplifying the test function technique we are going to apply in the sequel. Recall that $e_j := \{\pi_j(]0,l_j[)\}$, $j \in J$ and $\bar{e}_j := \{\pi_j([0,l_j])\}$, $j \in J$ denote the *open* and *closed* edges, respectively.

**Definition 5.1** *Let $\varphi \in C(\Gamma)$.*

*(i) Let $j \in J$ and $x \in e_j$. We say that $\varphi$ is* differentiable at $x$, *if $\varphi^j$ is differentiable at $\pi_j^{-1}(x)$.*

*(ii) Let $i \in I_T$ and $j,k \in \mathrm{Inc_i}$, $j \neq k$. We say that $\varphi$ is* $(j,k)$-differentiable at $v_i$, *if the one-sided differentials $\partial_j\varphi(v_i) := \partial_j\varphi^j(\pi_j^{-1}(v_i))$ and $\partial_k\varphi(v_i) := \partial_k(\varphi^k(\pi_k^{-1}(v_i))$ exist and satisfy*

$$a_{ij}\partial_j\varphi(v_i) + a_{ik}\partial_k\varphi(v_k) = 0,$$

*where $(a_{ij})$ is the oriented incidence matrix of $\Gamma$.*

**Definition 5.2** *Let $u, \varphi \in C(\Gamma)$.*

*(i) Let $j \in J$ and $x \in e_j$. We call $\varphi$ an* upper (lower) test function of $u$ at $x$, *if $\varphi$ is differentiable at $x$ and if $u - \varphi$ attains a local maximum (minimum) at $x$.*

*(ii) Let $i \in I_T$ and $j, k \in \mathrm{Inc_i}$. We call $\varphi$ an* upper (lower) $(j,k)$-test function at $v_i$, *if $\varphi$ is $(j,k)$-differentiable at $v_i$ and if $u - \varphi$ attains a local maximum (minimum) at $v_i$.*

**Remark.** Observe that if a function $f \in C(\Gamma)$ attains a local extremum at a vertex $v_i$, $i \in I_V$, this means that the value $u(v_i)$ is extremal with respect to the values $u(y)$ for all $y \in B_r(v_i)$ for some $r > 0$. The open ball $B_r(v_i) := \{y \in \Gamma \,|\, d(y, v_i) < r\}$, of course, is "star-shaped".

In order to simplify the notation let us from now on agree to shortly write $H^j(\partial_j u(x), v(x), x)$ for the expression $H^j(\partial_j u^j(\pi_j^{-1}(x)), v^j(\pi_j^{-1}(x)), \pi_j^{-1}(x))$ for all $x \in \bar{e}_j$, $j \in J$, whenever confusions are ruled out. We are then ready to introduce our notion of viscosity solutions on networks. Observe that the essentially new aspect is the condition (ii) at transition vertices.

**Definition 5.3** *Let $f : \Gamma \to \mathbb{R}$. Furthermore let $x \in \Gamma_0 := \Gamma \backslash \{v_i, i \in I_B\}$. A function $u \in C(\Gamma)$ is formally said to be a* viscosity sub- (super-) solution *of*

$$H(\partial u(x), u(x), x) = f(x) \tag{5.4}$$

*at $x$, if the following is the case.*

*(i) If $x \in e_j$, $j \in J$, we have*

$$H^j(\partial_j \varphi(x), u(x), x) \leq \ (\geq)\ f(x)$$

*for all upper (lower) test functions $\varphi \in C(\Gamma)$ of $u$ at $x$.*

*(ii) If $x = v_i$, $i \in I_T$, for each $j \in \mathrm{Inc_i}$ there is an index $k_1 \in \mathrm{Inc_i}$, $k_1 \neq j$, ($k_2 \in \mathrm{Inc_i}$, $k_2 \neq j$), such that*

$$H^j(\partial_j \varphi(x), u(x), x) = H^{k_l}(\partial_{k_l} \varphi(x), u(x), x) \leq \ (\geq)\ f(x), \quad l = 1\ (l = 2), \tag{5.5}$$

*for all upper (lower) $(j, k_l)$-test functions $\varphi \in C(\Gamma)$ at $v_i$. We call $k_l$ an* upper (lower) $i$-feasible index for $j$.

*Moreover, $u$ is said to be a* viscosity sub- (super) solution *of 5.4, if it is a viscosity sub- (super-) solution at each $x \in \Gamma_0$. (Alternatively, we say that $u$ solves $H(\partial u(x), u(x), x) \leq\ (\geq)\ f(x)$* in the viscosity sense.*) Furthermore, $u$ is said to be a* viscosity solution of 5.4 *(or it solves 5.4* in the viscosity sense*), if it is both a viscosity sub- and a supersolution.*

**Remark.** Observe that the first equality of 5.5 always holds due to 5.3 (vii). It is only stated to illustrate the symmetrical nature of this condition.

## 5.4 Uniqueness

The purpose of this chapter is to give a uniqueness proof for viscosity solutions on networks. In other words, we show uniqueness of solutions of the viscosity version of boundary value problem given by 5.1

$$\begin{aligned} &H(\partial u(x), u(x), x) = 0 && \text{on } \Gamma \text{ in the viscosity sense,} \\ &u(v_i) = g_i && \text{for all } i \in I_B. \end{aligned} \tag{5.6}$$

Before getting started, let us remark that the major part of the proof consists in establishing a comparison result, which can be considered to be the core of variety of different uniqueness proofs. Deriving uniqueness from this comparison result is then a matter of a straightforward adaptation of existing techniques in the literature. For the sake of completeness we decided to present an adaptation of a uniqueness proof due to H. Ishii, which is essentially based on the convexity assumption 5.2 (v).

In order to be able to state the comparison result we from now on assume an additional condition on the Hamiltonians $H^j$ to be true: For any fixed $z \in \mathbb{R}$ and $x \in [0, l_j]$, $j \in J$, we have

$$|\partial_j H^j(p, z, x)| \leq C < \infty \text{ for all } p \in \mathbb{R}. \tag{5.7}$$

**Lemma 5.1** *Let $f \in C(\Gamma)$ with $f(x) < 0$ for all $x \in \Gamma$ and suppose that we have two functions $u, v \in C(\Gamma)$ such that*

$$H(\partial u(x), u(x), x) \leq f(x) \quad \textit{and} \quad H(\partial v(x), v(x), x) \geq 0 \tag{5.8}$$

*on $\Gamma$ in the viscosity sense. Assume also*

$$u \leq v \quad \textit{on} \quad \partial\Gamma := \{v_i \,|\, i \in I_B\}.$$

*Then we have $u \leq v$ on $\Gamma$.*

**Proof.** Set $M := \max(||u||_\infty, ||v||_\infty)$, where $||\cdot||_\infty$ denotes the supremum norm on $\Gamma$, and choose a function $\beta \in C^\infty(\mathbb{R})$ with $0 \leq \beta \leq 1$, $\beta(0) = 1$, $\beta'(0) = 0$, and $\beta(x) = 0$ if $|x| \geq 1$. Moreover, recall that $d(\cdot, \cdot)$ denotes the path metric on $\Gamma$. For $\varepsilon > 0$ we then define the function

$$\Phi : \Gamma \times \Gamma \to \mathbb{R} \quad \text{by} \quad \Phi(x, y) := u(x) - v(y) + 5M\beta(\varepsilon^{-1} d(x, y)).$$

Assume now that there is a point $z \in \Gamma_0$ with $u(z) > v(z)$. By means of the function $\Phi$ we derive a contradiction to this assumption.

Note that $\Phi$ is continuous on the compact set $\Gamma \times \Gamma$. Therefore it attains its maximum at some point $(p, q) \in \Gamma^2$. Observe that the construction of $\Phi$ forces $p$ and $q$ to be not further apart than a distance of $\varepsilon$, i.e. $d(p, q) \leq \varepsilon$. Moreover, neither $p$ nor $q$ lie in $\partial\Gamma$, if $\varepsilon$ is sufficiently small. To see this, set

$$m_\varepsilon := \sup\{u(x) - v(y) \,|\, d(x, y) \leq \varepsilon, (x, y) \in \Gamma^2 \backslash \Gamma_0^2\}$$

and observe that $\limsup_{\varepsilon\to 0} m_\varepsilon \leq 0$ as $\varepsilon \to 0$ due to continuity reasons and to the fact that we have $u \leq v$ on $\partial\Gamma$. Now choose $\varepsilon > 0$ sufficiently small that we have $m_\varepsilon < u(z) - v(z)$. Then $\Phi(z,z) > \Phi(x,y)$ for any choice $(x,y) \in \Gamma^2 \backslash \Gamma_0^2$ with $d(x,y) \leq \varepsilon$. Consequently we have $(p,q) \in \Gamma_0^2$.

If necessary we further decrease $\varepsilon$ to arrange that we have $0 < u(x) - v(x) \leq u(z) - v(z)$ for all $x \in \Gamma$ with $d(x,z) \leq \varepsilon$, which is possible due to continuity reasons.

Now choose any $m \in \mathbb{R}$ with $0 < m < -\max_{x\in\Gamma} f(x)$ and any compact set $U \subset \mathbb{R}$ such that $u(x) \in U$ for all $x \in \Gamma$. Then choose $\varepsilon > 0$ small enough to ensure $\varepsilon C_0 \leq m/2$, where

$$C_0 := \max\{|\partial_j H^j(p,z,x)| \,|\, (p,z,x) \in \mathbb{R} \times U \times \, [0,l_j], j \in J\},$$

which is a finite number due to 5.7.

Finally we adjust $\varepsilon$ for a last time to ensure that there is a unique path of length $d(p,q)$ in $\Gamma$ which connects $p$ and $q$ and which runs through at most one vertex $v_i$, $i \in I_T$. Observe now that the situation may be described by one of the following different cases.

*Case* 1. In this case there are indices $i \in I_T$ and $j,k \in \mathrm{Inc_i}$ such that $p \in e_j$ and $q \in e_k$. Consider the functions $\varphi_q, \varphi_p \in C(\Gamma)$ defined by

$$\varphi_q : x \mapsto 5M\beta(\varepsilon^{-1} d_q(x)) \quad \text{and} \quad \varphi_p : x \mapsto 5M\beta(\varepsilon^{-1} d_p(x)),$$

where we shortly write $d_y := d(\cdot, y)$ for all $y \in \Gamma$. Clearly $\varphi_q^j$ and $\varphi_p^k$ are continuously differentiable at $\tilde{p} := \pi_j^{-1}(p)$ and $\tilde{q} := \pi_k^{-1}(q)$, respectively.

Now observe that by the choice of $p$ and $q$ the function $u + \varphi_q$ attains a local maximum at $p$, whence $-\varphi_q$ is an upper test function of $u$ at $p$. By 5.8 we thus deduce the following relation:

$$\begin{aligned} f(p) &> H^j(-\partial_j \varphi_q^j(\tilde{p}), u^j(\tilde{p}), \tilde{p}) = H^j(-5M\partial_j[\beta(\varepsilon^{-1} d_q^j)]|_{x=\tilde{p}}, u^j(\tilde{p}), \tilde{p}) \\ &= H^j(-5Ma_{ij}\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\tilde{p}), \tilde{p}), \end{aligned} \tag{5.9}$$

where we have used $\partial_j d_q^j(\tilde{p}) = a_{ij}$.

Similarly observe that $-v + \varphi_p$ attains a local maximum at $q$, implying that $v - \varphi_p$ attains a local minimum at $q$. Hence $\varphi_p$ is a lower test function of $v$ at $q$ and by 5.8 we conclude

$$\begin{aligned} 0 &\leq H^k(\partial_k \varphi_p^k(\tilde{q}), v^k(\tilde{q}), \tilde{q}) = H(5M\partial_k[\beta(\varepsilon^{-1} d_p^k)]|_{x=\tilde{q}}, v^k(\tilde{q}), \tilde{q}) \\ &= H^k(5Ma_{ik}\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), v^k(\tilde{q}), \tilde{q}). \end{aligned} \tag{5.10}$$

Observe that by 5.9 and by the definition of $m$ we have

$$0 > f(p) + m \geq H^j(-5Ma_{ij}\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\tilde{p}), \tilde{p}) + m =: T_1. \tag{5.11}$$

The relation $|\tilde{p} - \pi_j^{-1}(v_i)| = d(p, v_i) \leq \varepsilon$, the definition of $C_0$, and the relation $\varepsilon C_0 < m$ imply

$$T_1 \geq H^j(-5Ma_{ij}\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\tilde{p}), \pi_j^{-1}(v_i)) + m/2 =: T_2. \tag{5.12}$$

Using $|\tilde{q} - \pi_k^{-1}(v_i)| = d(q, v_i) \leq \varepsilon$ and the conditions 5.3 (vi) and (vii) we obtain

$$T_2 > H^k(5Ma_{ik}\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\tilde{p}), \tilde{q}) =: T_3 \tag{5.13}$$

By means of

$$u^j(\tilde{p}) - v^k(\tilde{q}) = u(p) - v(q) \geq u(z) - v(z) > 0$$

in combination with 5.2 (iii) we get

$$T_3 \geq H^k(5Ma_{ik}\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), v^k(\tilde{q}), \tilde{q}) \geq 0, \tag{5.14}$$

where the last inequality follows by 5.10. The combination of 5.11 - 5.14 then yields a contradiction. Note that the symmetry condition 5.2 (vi) is only required if $a_{ij} = a_{ik}$, that is, if both $e_j$ and $e_k$ are equally orientated with respect to $v_i$.

*Case 2.* In this case we assume $p \neq q$ and $p, q \in e_j$ for some $j \in J$. Observe that with $\tilde{p} := \pi_j^{-1}(p)$ and $\tilde{q} := \pi_j^{-1}(q)$ we then have

$$\partial_j d_q^j(\tilde{p}) = -\partial_j d_p^j(\tilde{q}) =: \sigma \in \{-1, 1\}.$$

Proceeding similarly to case 1 and using the functions $-\varphi_q$ and $\varphi_p$ as upper and lower test functions for $u$ and $v$ at $p$ and $q$, respectively, we obtain

$$H^j(-5M\sigma\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\tilde{p}), \tilde{p}) < f(p) \tag{5.15}$$

and

$$H^j(-5M\sigma\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), v^j(\tilde{q}), \tilde{q}) \geq 0. \tag{5.16}$$

As in case 1, this yields the contradiction

$$\begin{aligned} 0 > f(p) + m &\geq H^j(-5M\sigma\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\tilde{p}), \tilde{p}) + m \\ &\geq H(-5M\sigma\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\tilde{p}), \tilde{q}) + m/2 \\ &\geq H(-5M\sigma\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), v^j(\tilde{q}), \tilde{q}) \geq 0. \end{aligned}$$

*Case 3.* Now we assume that one of the points, say $p$, coincides with a transition vertex. Accordingly, we assume that there are indices $i \in I_T$ and $j \in \mathrm{Inc_i}$ such that $p = v_i$ and $q \in e_j$.

As in the previous cases we consider the functions $\varphi_p$ and $\varphi_q$. Taking into account that $p = v_i$, we observe that for all $k \in \mathrm{Inc_i}$ the one-sided differentials $d_k := \partial_k \varphi_q^k(\pi_k^{-1}(p))$ exist. From $q \in e_j$

it follows that for all $k \in \mathrm{Inc_i}$ with $k \neq j$ we have $a_{ik}d_k = 1$, whereas we have $a_{ij}d_k = -1$. Consequently,

$$a_{ij}\partial_j\varphi_q^j(\pi_k^{-1}(p)) + a_{ik}\partial_k\varphi_q^k(\pi_k^{-1}(p)) = 0 \tag{5.17}$$

for all $k \in \mathrm{Inc_i}$, $k \neq j$. Furthermore, $u + \varphi_q$ attains a local maximum at $p$. Hence, by definition 5.2 (ii), $-\varphi_q$ is an upper $(j,k)$-test function of $u$ at $p$ for all $k \in \mathrm{Inc_i}$, $k \neq j$. On the other hand, according to definition 5.3 there is an index $k \in \mathrm{Inc_i}$, $k \neq j$, such that we have

$$H^j(\partial_j\varphi^j(\pi_j^{-1}(p)), u^j(\pi_j^{-1}(p)), \pi_j^{-1}(p)) = H^k(\partial_k\varphi^k(\pi_k^{-1}(p)), u^k(\pi_k^{-1}(p)), \pi_k^{-1}(p)) < f(p) \tag{5.18}$$

for all upper $(j,k)$-test functions $\varphi$ of $u$ at $p$. Consequently we may plug $-\varphi_q$ into 5.18 and obtain by virtue of 5.3 (vi)

$$H^j(-5M\sigma\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), u^j(\pi_j^{-1}(p)), \pi_j^{-1}(p)) < f(p) \tag{5.19}$$

for both $\sigma = 1$ and $\sigma = -1$.

Observe next that $\varphi_p$ is continuously differentiable and attains a local minimum at $q$. Hence it is a lower test function of $v$ at $q$, and we derive as in case 2

$$H^j(5Ma_{ij}\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)), v^j(\pi_j^{-1}(q)), \pi_j^{-1}(q)) \geq 0.$$

Choosing $\sigma := -a_{ij}$ in 5.19 we now proceed as in case 2 to derive a contradiction.

*Case 4.* In this last case we assume that $p$ and $q$ coincide. If they coincide on a transition vertex $v_i$, $i \in I_T$, by the fact that we have $\beta'(0) = 0$ it follows

$$\partial_j\varphi_p^j(\pi_j^{-1}(v_i)) = \partial_j\varphi_q^j(\pi_j^{-1}(v_i)) = 0$$

for all $j \in \mathrm{Inc_i}$. In particular, for each choice of $j,k \in \mathrm{Inc_i}$, $j \neq k$, both $-\varphi_q$ and $\varphi_p$ are upper and lower $(j,k)$-test functions of $u$ and $v$ at $v_i$, respectively, and a contradiction similar to the previous cases can be derived.

The case $p = q \in e_j$, $j \in J$, can be treated analogously. □

As we have already mentioned, an extension of the comparison lemma 5.1 to a uniqueness result can be achieved by various methods, among which we have chosen to present the one given in H. Ishii [19], which is primarily based upon the convexity of the $H^j$ in the third argument (see 5.2 (v)).

**Lemma 5.2** *Let $u, v \in C(\Gamma)$ be a viscosity sub- and supersolution of*

$$H(\partial u(x), u(x), x) = 0 \quad on\ \Gamma, \tag{5.20}$$

*respectively, with $u(v_i) \leq v(v_i)$ for all $i \in I_B$. Then $u \leq v$ on $\Gamma$.*

**Proof.** Fix a number $M$ with $M < \min_{x\in\Gamma} u(x)$ and set

$$u_\theta(x) := \theta u(x) + (1-\theta)M, \quad x \in \Gamma,$$

for $\theta \in ]0,1[$. Clearly we have $u_\theta < u$ on $\Gamma$ and $u_\theta \in C(\Gamma)$. Now choose a compact set $K \subset \mathbb{R}$ such that $u_\theta(x) \in K$ for all $x \in \Gamma$ and $\theta \in [0,1]$. Due to conditions 5.2 (i) and 5.3 (vii) it is then possible to choose a function $h \in C(\Gamma)$ with $h < 0$ on $\Gamma$ and

$$H^j(0,z,x) \leq h^j(x) \tag{5.21}$$

for all $x \in [0,l_j]$, $j \in J$, $z \in K$, and for all $\theta \in [0,1]$. Using this, we show that we have

$$H(\partial u_\theta(x), u_\theta(x), x) \leq (1-\theta)h(x) \tag{5.22}$$

on $\Gamma$ in the viscosity sense. For this purpose fix $j \in J$, $x \in e_j$, and set $\tilde{x} := \pi_j^{-1}(x)$. Let $\varphi \in C(\Gamma)$ be an upper test function of $u$ at $x$. Setting $\varphi_\theta := \theta\varphi + (1-\theta)M$ we obtain by means of the convexity condition 5.3 (v)

$$\begin{aligned} & H^j(\partial_j\varphi_\theta^j(\tilde{x}), u_\theta^j(\tilde{x}), \tilde{x}) \leq \theta H^j(\partial_j\varphi^j(\tilde{x}), u_\theta^j(\tilde{x}), \tilde{x}) + (1-\theta)H^j(0, u_\theta^j(\tilde{x}), \tilde{x}) \\ \leq \ & \theta H^j(\partial_j\varphi^j(\tilde{x}), u^j(\tilde{x}), \tilde{x}) + (1-\theta)h^j(\tilde{x}). \end{aligned} \tag{5.23}$$

We have also employed 5.21 as well as the relation $u_\theta^j(\tilde{x}) < u^j(\tilde{x})$ in combination with condition 5.2 (iii). As $u$ is a viscosity subsolution and as $\varphi$ is an upper test function of $u$ at $x$, we have

$$H^j(\partial_j\varphi^j(\tilde{x}), u^j(\tilde{x}), \tilde{x}) \leq 0,$$

whence we further conclude from 5.23

$$H^j(\partial_j\varphi_\theta(\tilde{x}), u_\theta^j(\tilde{x}), \tilde{x})) \leq (1-\theta)h^j(\tilde{x}).$$

Secondly, assume $x = v_i$ for some $i \in I_T$. As $u$ is a viscosity subsolution, for $j \in \mathrm{Inc_i}$ there is an upper $i$-feasible index $k \in \mathrm{Inc_i}$, that is, all upper $(j,k)$-test functions $\varphi \in C(\Gamma)$ of $u$ at $x$ satisfy

$$H^j(\partial_j\varphi^j(\pi_j^{-1}(v_i)), u^j(\pi_j^{-1}(v_i)), \pi_j^{-1}(v_i)) = H^k(\partial_k\varphi^k(\pi_k^{-1}(v_i)), u^k(\pi_k^{-1}(v_i)), \pi_k^{-1}(v_i)) \leq 0.$$

Hence, assuming $\varphi$ to be an upper $(j,k)$-test function of $u$ at $x$ and performing the same calculation as above yields

$$H^j(\partial_j\varphi_\theta^j(\pi_j^{-1}(v_i)), u_\theta^j(\pi_j^{-1}(v_i)), \pi_j^{-1}(v_i)) \leq (1-\theta)h^j(\pi_j^{-1}(v_i)).$$

Hence 5.22 is shown.

Now observe that we have $u_\theta \leq v$ on $\partial\Gamma$. As $v$ is a viscosity supersolution of 5.20 and by taking advantage of 5.22 we now apply lemma 5.1 with $f := (1-\theta)h$ to conclude that we have $u_\theta \leq v$ for all $\theta \in ]0,1[$. Letting $\theta$ tend to 1 completes the proof. □

## 5.5 Existence

Having established the uniqueness of viscosity solutions on networks, we now turn to the question of their existence. The idea is to apply a Perron method to construct a viscosity solution. In fact, it is given by the pointwise supremum over the class of all those viscosity subsolutions of 5.6, which are smaller than or equal to the prescribed boundary values at the boundary vertices – we refer to these functions as *subfunctions*. The proof consists of two parts: first it has to be shown that the pointwise supremum over the class of all subfunctions is a subfunction itself, and then it has to be verified that it also is a viscosity supersolution. The latter case is treated indirectly, by assuming it not to be a supersolution and concluding that in this case one can construct a strictly larger subfunction, contradicting to the pointwise supremal property.

Accordingly, we start with the following general lemma ensuring the preservation of the subsolution property for pointwise suprema. Let us denote by $\underline{\mathrm{S}}$ the set of all subfunctions of 5.6.

**Lemma 5.3** *Let $V \subseteq \underline{S}$ and $u(x) := \sup_{v \in V} v(x)$ for all $x \in \Gamma$. Then we have $u \in \underline{S}$.*

**Proof.** The task of the proof is to verify the viscosity subsolution condition for $u$ at each point $x \in \Gamma_0$. We will, however, only treat the case $x = v_i$, $i \in I_T$, in detail, as the simpler case $x \in e_j$, $j \in J$, is essentially based upon the same arguments.

Accordingly, let $x = v_i$, $i \in I_T$. Fix any $j \in \mathrm{Inc_i}$. As $u$ is the pointwise supremum over $V$, for each $m \in \mathbb{N}$ there is a subfunction $u_m \in V$ with $u(x) - u_m(x) \leq 1/m$. For each $m \in \mathbb{N}$ we can find an $i$-feasible index $k(m) \in \mathrm{Inc_i}$ for $j$, as $u_m$ is a viscosity subsolution of 5.6. As $|\mathrm{Inc_i}| < \infty$, there is at least one $k \in \mathrm{Inc_i}$, $k \neq j$, such that $k(m_l) = k$ for a strictly increasing sequence $m_l$, $l \in \mathbb{N}$, of natural numbers. For reasons of simplicity we relabel $u_l := u_{m_l}$. Note that we have $u(x) - u_l(x) \leq 1/m_l < 1/l$.

We now show that $u$ solves 5.6 at $x$ in the viscosity sense. Accordingly, suppose that $\varphi \in C(\Gamma)$ be an upper $(k,l)$-test function of $u$ at $x$. We have to show

$$H^j(\partial_j \varphi(x), u(x), x) = H^k(\partial_k \varphi(x), u(x), x) \leq 0. \tag{5.24}$$

For this purpose we introduce the auxiliary function

$$\varphi_\delta : \Gamma \to \mathbb{R} \quad \text{with } \varphi_\delta(y) := \varphi(y) + \delta(d(x,y))^2,\ \delta > 0.$$

By the fact that $\varphi$ is an upper $(j,k)$-test function at $x$ we conclude that $u - \varphi_\delta$ attains a local maximum at $x$. This maximum is not only attained with respect to $\Gamma$ but also with respect to $\Theta := \bar{e}_j \cup \bar{e}_k$. Thus there is a radius $r > 0$ such that $u(y) - \varphi_\delta(y) \leq u(x) - \varphi_\delta(x)$ for all $y \in B_r(x) \cap \Theta$. For any $l \in \mathbb{N}$ the continuous function

$$\psi_l : y \mapsto u_l(y) - \varphi(y) - 2\delta(d(x,y))^2,$$

restricted to the compact set $\bar{B}_r(x)\cap\Theta$, attains its maximum at some point $x_l \in \bar{B}_r(x)\cap\Theta$. We claim that the sequence $(x_l)_{l\in\mathbb{N}}$ converges to $x$ as $l\to\infty$. To see this, we first invoke the fact that the $x_l$ are maximal points of the $\psi_l$, yielding

$$u_l(x) - \varphi(x) \le u_l(x_l) - \varphi(x_l) - 2\delta(d(x,x_l))^2.$$

Using $u(x) - u_l(x) < 1/l$ along with $u_l(x_l) \le u(x_l)$ we further conclude

$$u(x) - 1/l < u(x_l) + \varphi(x) - \varphi(x_l) - 2\delta(d(x,x_l))^2. \tag{5.25}$$

On the other hand, by the fact that $u-\varphi_\delta$ attains a local maximum at $x$, we obtain

$$u(x) - \varphi(x) \ge u(x_l) - \varphi(x_l) - \delta(d(x,x_l))^2. \tag{5.26}$$

Subtracting inequality 5.26 from 5.25 yields

$$-1/l + \varphi(x) < \varphi(x) - \delta(d(x,x_l))^2 \quad \text{or} \quad 1/l > \delta(d(x,x_l))^2,$$

implying $x_l \to x$ as $l\to\infty$. This, in turn, implies that for sufficiently large $l\in\mathbb{N}$ the local maxima of the functions $\psi_l$ at $x_l$ are attained with respect to the whole of $\Theta$. This means that the functions $u_l - \varphi_{2\delta}$ attain local maxima at $x_l$, if $l$ is sufficiently large, where

$$\varphi_{2\delta}(y) := \varphi(y) + 2\delta(d(x,y))^2.$$

Furthermore, $\varphi_{2\delta}$ respectively is differentiable or $(j,k)$-differentiable at $x_l$, if $x_l\neq x$ or $x_l = x$. In the latter case we remark that we indeed have the required property

$$a_{ij}\partial_j\varphi_{2\delta}(x) + a_{ik}\partial_k\varphi_{2\delta}(x) = 0,$$

as we have

$$\partial_j\varphi_{2\delta}(x) = \partial_k\varphi_{2\delta}(x) = 0.$$

For sufficiently large $l\in\mathbb{N}$ the function $\varphi_{2\delta}$ therefore respectively is an upper test function or an upper $(j,k)$-test function of $u_l$ at $x_l\in\Theta$, if $x_l\neq x$ or $x_l = x$. Using the fact that $u_l$ is a viscosity subsolution of 5.6 we therefore conclude

$$0 \ge H^s(\partial_s\varphi_{2\delta}(x_l), u_l(x_l), x_l) = H^s(\partial_s\varphi(x_l) + 4\delta a_{is}d(x,x_l), u_l(x_l), x_l) \tag{5.27}$$

if $x_l \in x_s(]0,l_s[)$ for either $s=l$ or $s=k$, and

$$H^j(0,x,u_l(x)) = H^k(0,x,u_l(x)) \le 0, \tag{5.28}$$

if $x_l = x$.

As $\varphi$ is $(j,k)$-differentiable at $x$ and as $x_l \to x$, the continuity of $H$ and $u$ implies that 5.27 and 5.28 yield

$$H^j(\partial_{x_j}\varphi(x), u(x), x) = H^k(\partial_{x_k}\varphi(x), u(x), x) \leq 0$$

upon letting $l \to \infty$. Hence we have shown that $k$ is an upper $i$-feasible index for $j$. As $j \in \mathrm{Inc_i}$ has been chosen arbitrarily, it follows that $u$ satisfies the viscosity subsolution property at $x$. Hence $u$ is a subfunction, as we clearly have $u(v_i) \leq \lim_{l\to\infty} u_l(v_i) \leq g_i$ for all $i \in I_B$. □

Now let $X$ be the set of all subfunctions of 5.6. The preceding lemma shows that the function $\bar{u}(x) := \sup_{u\in X} u(x)$, $x \in \Gamma$, is a subfunction itself, i. e. $\bar{u} \in X$. Accordingly, two questions remain: first, if $\bar{u}$ is a viscosity supersolution, and second, if it attains the boundary values. The first question is answered by the following lemma.

**Lemma 5.4** *The function $\bar{u}$ is a viscosity supersolution of 5.6.*

**Proof.** We present an indirect proof according to the following principle: If some function $u \in X$ does *not* satisfy the supersolution property at some point $y \in \Gamma_0$, we show that there is a function $v \in X$ with $v(y) > u(y)$. Conversely it follows that the pointwise supremum $\bar{u}$ must be a viscosity supersolution, as it is contained in $X$ by lemma 5.3.

Accordingly, let $u \in X$ and assume it not to satisfy the supersolution condition at some point $y \in \Gamma_0$. (Again we restrict ourselves to the case $y = v_i$, $i \in I_T$, as the simpler case $y \in e_j$, $j \in J$, can be easily derived using the same arguments.) Then there is an index $j \in \mathrm{Inc_i}$ for which there does *not* exist any lower $i$-feasible index $k \in \mathrm{Inc_i}$, $k \neq j$. By definition 5.3 this implies that for each $k \in \mathrm{Inc_i}$, $k \neq j$, there is a lower $(j,k)$-test function $\varphi_k$ of $u$ at $y$ with

$$H^j(\partial_j\varphi_k(y), u(y), y) = H^k(\partial_k\varphi_k(y), u(y), y) < 0. \tag{5.29}$$

Observe that without loss of generality we may assume that we have

$$\varphi_k(y) = u(y) \tag{5.30}$$

for all $k \in \mathrm{Inc_i}$, $k \neq j$. Furthermore, as for each of these $k$ the function $\varphi_k$ is a lower $(j,k)$-test function of $u$ at $y$, the function $u - \varphi_k$ attains a local minimum at $y$. Note that we may assume these minima to be *strict* by possibly adding to each $\varphi_k$ a parabola of the form $x \to \alpha_k(d(x,y))^2$ for some $\alpha_k < 0$. Then there is a $t > 0$ with $\bar{B}_t(y) \subset \Gamma_0$ such that $0 < (u - \varphi_k)(x)$ for all $x \in \bar{B}_t(y)\backslash\{y\}$ and such that for all $k \in \mathrm{Inc_i}\backslash\{\mathrm{l}\}$ – due to 5.29, 5.30, and to the continuity of both $H$ and $\varphi_k$ – the inequalities

$$H^j(\partial_j\varphi_k(x), \varphi_k(x), y) < 0 \quad \text{and} \quad H^k(\partial_k\varphi_k(x), \varphi_k(x), y) < 0 \tag{5.31}$$

are preserved for all $x \in \bar{B}_t(y) \cap e_l$. In particular, there is a $\xi > 0$ such that $u(x) - \varphi_k(x) > \xi$ for all $k \in \mathrm{Inc_i}$, $k \neq j$, and all $x \in \Gamma$ with $d(x,y) = t$. We furthermore can assume $\xi$ to be small enough such that the relations

$$H^j(\partial_j\varphi_k(x), \varphi_k(x) + \xi, y) < 0 \quad \text{and} \quad H^k(\partial_k\varphi_k(x), \varphi_k(x) + \xi, y) < 0 \tag{5.32}$$

hold for all $k \in \mathrm{Inc_i}\backslash\{\mathrm{j}\}$, which is possible by the continuity of $H$ and by relation 5.31. Hence it follows that for

$$\tilde{\varphi}_k := \varphi_k + \xi$$

we have $\tilde{\varphi}_k(y) = u(y) + \xi > u(y)$ and

$$\tilde{\varphi}_k(x) < u(x) \tag{5.33}$$

for all $k \in \mathrm{Inc_i}\backslash\{\mathrm{j}\}$ and all $x \in \Gamma$ with $d(x, y) = t$.

We now define the function

$$v(x) := \begin{cases} \max\left\{\max_{k\in\mathrm{Inc_i}\backslash\{\mathrm{j}\}} \tilde{\varphi}_k(x), u(x)\right\} & \text{if } x \in \bar{B}_t(y) \cap \bar{e}_j \\ \max\left\{\tilde{\varphi}_k(x), u(x)\right\} & \text{if } x \in \bar{B}_t(y) \cap e_k,\ k \in \mathrm{Inc_i}\backslash\{\mathrm{j}\} \\ u(x) & \text{if } x \in \Gamma\backslash\bar{B}_t(y) \end{cases}$$

and collect properties: By 5.33 we have $v \in C(\Gamma)$. Furthermore, $v$ is larger than $u$ at $y$, as we have $v(y) = u(y) + \xi > u(y)$. Let us demonstrate that $v$ is a viscosity subsolution, i.e. $v \in X$. For this purpose we first note that outside of $\bar{B}_t(y)$ we have $v \equiv u$, whence it suffices to check the viscosity subsolution property of $v$ for all $x \in \bar{B}_t(y)$. We have to distinguish two cases.

*Case 1.* Let $x \in \bar{B}_t(y) \cap e_l$ for some $l \in \mathrm{Inc_i}$ and let $\psi$ be an upper test function of $v$ at $x$. From 5.32 it follows that for each $k \in \mathrm{Inc_i}$, $k \neq j$, the function $\tilde{\varphi}_k$ satisfies

$$H^l(\partial_l\tilde{\varphi}_k(x), \tilde{\varphi}_k(x), x) < 0.$$

Consequently, as by definition $\tilde{\varphi}_k$ is continuously differentiable at $x$, this relation immediately implies the viscosity subsolution property of $\tilde{\varphi}_k$ at $x$ – a fact which can easily be verified. At $x$ it therefore follows that $v$ is the maximum of finitely many functions which all have the viscosity subsolution property at $x$. Now observe that for those of these functions, which attain the (maximal) value $v(x)$ at $x$, the function $\psi$ forms an upper test function at $x$ and thus inherits the property

$$H^l(\partial_l\psi(x), v(x), x) \leq 0.$$

Hence $v$ has the viscosity subsolution property at $x$.

*Case 2.* We now derive the viscosity subsolution property of $v$ at $y = v_i$. To this end we have to show that we can assign an upper $i$-feasible index to each index $k \in \mathrm{Inc_i}$. In fact, as far as the index $j$ is concerned, any choice of $k \in \mathrm{Inc_i}\backslash\{\mathrm{j}\}$ yields an upper $i$-feasible index for $j$. For, fixing such an index $k$ and letting $\psi$ be an upper $(j,k)$-test function of $v$ at $y$, by construction of $v$ we have $\tilde{\varphi}_k(y) = v(y)$. This implies that $\psi$ also is an upper $(j,k)$-test function of $\tilde{\varphi}_k$ at $y$. On the other hand, $\tilde{\varphi}_k$ is $(j,k)$-differentiable at $y$ and satisfies

$$H^j(\partial_j\tilde{\varphi}_k(y), \tilde{\varphi}_k(y), y) = H^k(\partial_k\tilde{\varphi}_k(y), \tilde{\varphi}_k(y), y) < 0,$$

which follows by inequalities 5.32 and 5.29. This, in turn, implies

$$H^j(\partial_j \psi(y), v(y), y) = H^k(\partial_k \psi(y), v(y), y) \leq 0,$$

whence $k$ is an upper $i$-feasible index for $j$.

It remains to remark that the same argumentations show that the index $j$ itself serves as an upper $i$-feasible index for each index $k \in \mathrm{Inc_i} \backslash \{\mathrm{j}\}$, by symmetry. It follows that $v$ has the viscosity subsolution property at $y$. We thus have shown that $v$ is contained in $X$ and is strictly larger than $u$ at $y$, which completes the proof. □

## 5.6 Consistency with vanishing viscosity

The purpose of the present section is to verify the consistency of viscosity solutions on networks with the method of vanishing viscosity, the convergence of which has been shown in chapter 4. As has been mentioned before, the introduction of viscosity solutions on networks will only be fully justified as a correct generalization of classical viscosity solutions, when we will have shown that the limit function of the vanishing viscosity method indeed is a viscosity solution.

Let us therefore revisit the boundary value problem 4.7, where the Hamiltonians $H^j$ again are assumed to satisfy the conditions 4.8 and 4.9. Theorem 4.2 states that the functions $u_\varepsilon$ uniformly converge to a limit function $u \in C(\Gamma)$, as $\varepsilon \to 0$. Let us here and in the sequel assume that the functions $u_\varepsilon$ and their first derivatives be uniformly bounded in $\varepsilon$. Then it will be confirmed by the following lemmas that $u$ indeed is a viscosity solution of

$$H(\partial u(x), u(x), x) = 0$$

on $\Gamma$ in the sense of definition 5.3.

**Lemma 5.5** *Let $i \in I_T$ and $j \in \mathrm{Inc_i}$. Furthermore let the sequence $x_m \in e_j$ converge to $v_i$. Then we have*

$$H^j(p_0, u(v_i), v_i) \leq 0,$$

*where*

$$p_0 := \limsup_{m\to\infty} \left| \frac{u(x_m) - u(v_i)}{d(x_m, v_i)} \right|.$$

**Proof.** Suppose without loss of generality that we have $a_{ij} = 1$ and $u(v_i) = u^j(0) = 0$. Setting $v := u^j$ and $y_m := \pi_j^{-1}(x_m)$ we then have to show that we have

$$h(\limsup_{m\to\infty} |v(y_m)/y_m|) \leq 0,$$

where we define

$$h(p) := H^j(p, 0, v_i), \quad p \in \mathbb{R}.$$

For this purpose first observe that by 4.8 and 4.9 the function $h$ is twice continuously differentiable and strictly convex on $\mathbb{R}$ as well as symmetric in $p$. Furthermore $h$ is strictly increasing on $[0, \infty[$ and strictly decreasing on $]-\infty, 0]$.

Now set $v_\varepsilon := u_\varepsilon^j$, plug $y_m x$ into 4.7 and integrate in order to obtain

$$\int_0^1 H^j(\partial_j v_\varepsilon(y_m x), v_\varepsilon(y_m x), y_m x)\, dx = \varepsilon \int_0^1 \partial_j^2 v_\varepsilon(y_m x)\, dx.$$

Moreover we have

$$\left| \int_0^1 \partial_j^2 v_\varepsilon(y_m x)\, dx \right| \le 2 \max_{x \in [0, y_m]} |\partial_j v_\varepsilon(x)| < 2C,$$

where $C \ge 0$ is the uniform bound of the first derivatives of the functions $u_\varepsilon$, the existence of which has been assumed above. It follows that we have

$$\lim_{\varepsilon \to 0} \int_0^1 H^j(\partial_j v_\varepsilon(y_m x), v_\varepsilon(y_m x), y_m x)\, dx \to 0 \tag{5.34}$$

for all $m \in \mathbb{N}$.

Observe now that as $h$ is convex we may apply Jensen's inequality to compute

$$\begin{aligned} h\left(\frac{v_\varepsilon(y_m) - v_\varepsilon(0)}{y_m}\right) &= h\left(\frac{1}{y_m} \int_0^{y_m} \partial_j v_\varepsilon(x)\, dx\right) = h\left(\int_0^1 \partial_j v_\varepsilon(y_m x)\, dx\right) \\ &\le \int_0^1 h(\partial_j v_\varepsilon(y_m x))\, dx \le \int_0^1 H^j(\partial_j v_\varepsilon(y_m x), v_\varepsilon(y_m x), y_m x)\, dx + C y_m \end{aligned}$$

for some constant $C < \infty$, where the last inequality follows by a Taylor expansion of $H^j$ in the second and third argument as well as by the boundedness of $v_\varepsilon$ and $\partial_j v_\varepsilon$ uniformly in $\varepsilon$. Letting $\varepsilon \to 0$ yields by 5.34

$$h\left(\frac{v(y_m)}{y_m}\right) \le C y_m$$

and letting $m \to \infty$ implies

$$\limsup_{m \to \infty} h\left(\frac{v(y_m)}{y_m}\right) \le 0.$$

By the properties of $h$ it then follows

$$h\left(\limsup_{m \to \infty} \left|\frac{v(y_m)}{y_m}\right|\right) \le 0,$$

which completes the proof. $\square$

**Lemma 5.6** *The limit function $u$ satisfies $H(\partial u(x), u(x), x) \leq 0$ in the viscosity sense.*

**Proof.** The proof is divided into two parts: the first shows the subsolution property at points $x \in e_j$, $j \in J$, whereas it is verified at transition vertices in the second part.

*Part 1.* Let $j \in J$ and $x \in e_j$. Furthermore, let $\varphi$ be an upper test function of $u$ at $x$. Choose a sequence $\varepsilon_n \to 0$ and set $u_n := u_{\varepsilon_n}$, $n \in \mathbb{N}$. Now assume without loss of generality that the maximum of $u - \varphi$ attained at $x$ be *strict*, which we can always arrange by possibly adding a parabola centered at $x$. Furthermore, by means of density arguments, we can also assume $\varphi$ to be twice continuously differentiale around $x$. As the function $u_n$ uniformly converge to $u$, it then follows that there is a sequence $x_n \in \Gamma$, $n \in \mathbb{N}$, such that $x_n \to x$; without loss of generality we can assume $x_n \in e_j$ for all $n \in \mathbb{N}$. By the maximal property of the points $y_n := \pi_j^{-1}(x_n) \in ]0, l_j[$ we then have

$$\partial_j u_n^j(y_n) = \partial_j \varphi^j(y_n) \quad \text{and} \quad \partial_j^2 u_n^j(y_n) \leq \partial_j^2 \varphi^j(y_n)$$

for all $n \in \mathbb{N}$. It follows

$$\begin{aligned} 0 &= \varepsilon_n \partial_j^2 u_n^j(y_n) - H^j(\partial_j u_n^j(y_n), u_n^j(y_n), y_n) \\ &\leq \varepsilon_n \partial_j^2 \varphi^j(y_n) - H^j(\partial_j \varphi^j(y_n), u_n^j(y_n), y_n). \end{aligned}$$

Setting $y := \pi_j^{-1}(x)$ we obtain by $\lim_{n\to\infty} y_n = y$ and by the continuity of $H^j$

$$0 \leq \lim_{n\to\infty} \left[\varepsilon_n \partial_j^2 \varphi^j(y_n) - H^j(\partial_j \varphi^j(y_n), u_n^j(y_n), y_n)\right] = -H^j(\partial_j \varphi^j(y), u^j(y), y),$$

which proves the assertion.

*Part 2.* We now verify the viscosity subsolution property at transition vertices. For this purpose we fix $i \in I_T$ and $j \in \mathrm{Inc_i}$ and show that each index $k \in \mathrm{Inc_i}\backslash\{\mathrm{j}\}$ is upper $i$-feasible index for $j$. Accordingly, fix such an index $k$ and let $\varphi \in C(\Gamma)$ be an upper $(j,k)$-test function of $u$ at $v_i$. We then have to show that we have

$$H^j(\partial_j \varphi(v_i), u(v_i), v_i) = H^k(\partial_k \varphi(v_i), u(v_i), v_i) \leq 0.$$

To this end assume the contrary. Hence for one of the indices $j$ and $k$, say, for $j$, we have

$$a_{ij} \partial_j \varphi(v_i) < -a,$$

with $a > 0$ as defined in the proof of lemma 5.5. Assuming $u(v_i) = 0$ without loss of generality, it follows that we have for every sequence $x_m$, $m \in \mathbb{N}$, with $x_m \in e_j$ and $x_m \to v_i$

$$p_0 := \limsup_{m\to\infty} \frac{u(x_m) - u(v_i)}{d(x_m, v_i)} < -a,$$

as $u - \varphi$ attains a local maximum at $v_i$. By the properties of $H$, however, we then conclude

$$H^j(p_0, 0, v_i) > 0,$$

a contradiction to lemma 5.5. □

Now we show that $u$ is also a viscosity supersolution.

**Lemma 5.7** *The limit function $u := \lim_{\varepsilon\to 0} u_\varepsilon$ satisfies $H(x, u(x), Du(x)) \geq 0$ in the viscosity sense everywhere on $\Gamma$.*

**Proof.** Again the proof splits into two parts, the first one treating edge points, the second one treating transition vertices. However, the case of edge points can be performed perfectly analogously to the first part in the proof of the previous lemma, which is why we dispense with its presentation here.

The case of transition vertices, however, requires a different approach. In particular, the Kirchhoff condition will be invoked. Let $i \in I_T$ and assume $u(v_i) = 0$ without loss of generality. It then suffices to show that there is an index $j \in \mathrm{Inc_i}$ (for which we assume $a_{ij} = 1$ without loss of generality) such that for all functions $\psi \in C([0, l_j])$ for which $u^j - \psi$ attains a one-sided local minimum at 0 and for which the derivative $\partial_j\psi(0)$ exists, we have

$$\partial_j\psi(0) \leq -a.$$

Here, as in the proof of the previous lemma, $a > 0$ is the unique positive number satisfying

$$h(a) := H^j(a, 0, 0) = H^j(-a, 0, 0) = 0.$$

This is indeed sufficient, as we then in particular have

$$\partial_j\varphi^j(0) \leq -a$$

for all lower $(j, k)$-test functions $\varphi$ and any $k \in \mathrm{Inc_i}\backslash\{\mathrm{j}\}$, implying, by the properties of $H$, that we have

$$H^j(\partial_j\varphi(v_i), 0, v_i) = H^k(\partial_k\varphi(v_i), 0, v_i) \geq 0,$$

under the assumption $\varphi(v_i) = 0$. Hence $j$ is a lower $i$-feasible index for each $k \in \mathrm{Inc_i}\backslash\{\mathrm{j}\}$, and, by symmetry, any such $k$ is a lower $i$-feasible index for $j$, whence the assertion is proved.

Let us therefore show that such an index $j \in \mathrm{Inc_i}$ exists. First note that the functions $u_\varepsilon$, $\varepsilon > 0$, satisfy the Kirchhoff condition at $v_i$. We therefore may extract from each sequence $\varepsilon_n \to 0$ a subsequence (which we again denote by $\varepsilon_n$) such that there is an index $j \in \mathrm{Inc_i}$ with

$$\partial_j u_n^j(0) \leq 0 \tag{5.35}$$

for $u_n := u_{\varepsilon_n}$, where we have assumed that we have $a_{ij} = 1$ without loss of generality. Furthermore we *assume* to have $L^1$-convergence in the following sense

$$\lim_{n\to\infty} \int_0^{l_j} \left| H^j(\partial_j u_n^j(y), u_n^j(y), y) \right| \, dy = 0.$$

(entsprechendes Lemma wird noch geliefert).

It follows

$$\lim_{n\to\infty} H^j(\partial_j u_n^j(y), u_n^j(y), y) \to 0 \quad \text{for each } y \in M := [0, l_j] \backslash N, \tag{5.36}$$

where $N$ is a set of measure zero. Now the Taylor expansion of $H^j$ yields

$$h(\partial_j u_n^j(y)) = H^j(\partial_j u_n^j(y), 0, 0) \geq H^j(\partial_j u_n^j(y), u_n^j(y), y) - C_0 y \tag{5.37}$$

for each $y \in [0, l_j]$, where the constant $C_0 > 0$ can be chosen independently of $n \in \mathbb{N}$, as the functions $u_n$ along with their first derivatives are uniformly bounded in $n$.

By 5.36 and 5.37 it then follows

$$h\left(\lim_{n\to\infty} \partial_j u_n^j(y)\right) = \lim_{n\to\infty} h(\partial_j u_n^j(y)) \geq -C_0 y,$$

which implies by the properties of $h$ that there is a constant $C_1 > 0$ such that we have

$$\lim_{n\to\infty} |\partial_j u_n^j(y)| \geq a - C_1 y$$

for each $y \in M$. Now observe that the functions $u_n^j$, $n \in \mathbb{N}$, do not attain local minima on $]0, l_j[$, which is a direct consequence of 4.7. Hence we further conclude

$$\lim_{n\to\infty} \partial_j u_n^j(y_n) \leq -a \tag{5.38}$$

for all sequences $y_n \in M\backslash\{0\}$, $n \in \mathbb{N}$, $y_n \to 0$. Now assume that there is a test function $\psi \in C^2([0, l^j])$ such that $u^j - \psi$ attains a right-sided local minimum at 0 and such that

$$\partial_j \psi(0) \geq -a + \delta$$

for some $\delta > 0$. Again we can assume the minimum to be strict without loss of generality. As the functions $u_n$ uniformly converge to $u$, there is a sequence $\tilde{y}_n \to 0$, $\tilde{y}_n \geq 0$ for all $n \in \mathbb{N}$, such that $u_n^j - \psi$ attains a local minimum at $\tilde{y}_n$. By continuity reasons it then follows that there is a sequence $y_n \to 0$, $y_n \in M\backslash\{0\}$, such that we have

$$\partial_j \psi(y_n) \leq \partial_j u_n^j(y_n) + \delta/4$$

for all $n \in \mathbb{N}$. As $y_n \to 0$, we furthermore have

$$\partial_j \psi(y_n) \geq -a + \delta/2$$

for all sufficiently large $n \in \mathbb{N}$ due to continuity reasons. By 5.38 we then get the contradiction

$$-a + \delta/2 \leq \lim_{n\to\infty} \partial_j \psi(y_n) = \lim_{n\to\infty} \partial_j u_n^j(y_n) + \delta/4 \leq -a + \delta/4.$$

$\square$

# Chapter 6

# The distance function on networks

**Summary.** In the present chapter we revisit the distance function on networks and derive a topological property of its curvature functional. For illustrating reasons we connect the distance function to the problem of mathematical modelling of granular matter, where the curvature functional corresponds with the number of “hilltops” of sand heaps. We also discuss connections to abstract graph theory and shortest path algorithms appearing in computer scientifical contexts.

## 6.1 Introduction

The behavior of homogeneous granular matter, which is poured onto objects from sources above, is rather well understood and has been described by various models. Several one- and two-dimensional models have been proposed by Hadeler et al ([20], [21], [1]). In particular, situations have been examined where a maximal amount of sand has been deposited on top of a given object or domain. Dry and homogeneous granular matter can only form heaps whose local steepness does not exceed (the tangent of) the so-called angle of repose $\alpha$, which is specific to the respective material. As a consequence, the shape of maximal volume heaps on flat and bounded domains (such as tables) is approximately described by the graph of the unique viscosity solution $u$ (in the classical sense) of the eikonal equation $|\nabla u| = \tan\alpha$. The function $u$ has a gradient of constant length $\tan\alpha$ and of course coincides with a multiple of the distance function to the boundary.

Having established the notion of viscosity solution on topological networks in the previous chapter, it suggests itself to look for an appropriate granular matter interpretation of the eikonal equation on such a network. In this chapter we therefore want to discuss the shape of “sand heaps” formed by a maximal amount of granular matter within the framework of topological networks. This approach directly leads to the question about a certain characteristicum of those “heaps”: the number of “hilltops” or “points”. In fact, the core of the chapter will be the derivation of a simple formula for this number. This result generalizes the one-dimensional version of a result of Aviles and Giga ([2], lemma 2.6), basically stating that the suitably defined curvature functional of the distance function on an $n$-dimensional domain $\Omega$ equals the $n-1$-dimensional Hausdorff measure of the boundary $\partial\Omega$.

## 6.2 Granular matter and networks

Imagine a topological network $\Gamma = \Gamma(E, V)$ in the plane and think of its edges $e_j$, $j \in J$, as a set of slim "paths" whose end points are connected at the vertices $v_i$, $i \in I$. Let us suppose the paths on both sides to be bounded by thin, sufficiently high glass walls perpendicular to the plane, a construction we will from now on refer to as a "maze". As before we split the vertex set into a nonempty set of boundary vertices $\partial\Gamma = \{v_i, i \in I_B\}$ and a set of transition vertices $\{v_i, i \in I_T\}$, assuming that sand may run out of the maze at boundary points through a little hole at the bottom. Let us then uniformly pour as much sand as possible from above into the space between the glass walls. Several sand heaps will grow, every two of them separated by at least one boundary point, at which sand is running out of the maze. At transition vertices, sand may be interchanged between the different incident paths. As the distance between the two walls is very small, we may (and shall) consider the shape of the heaps as one-dimensional. Finally the shape of the sand heaps will stop growing and reach an equilibrium configuration, as additional sand poured on top of them will immediately leave the maze at boundary points. This is due to the fact that each newly added sand portion would locally violate the angle of repose.

Similarly to the case of intervals and two-dimensional domains it is easy to understand that in this maze model the resulting shape of the maximal volume heap is appropriately modelled by the graph of (a multiple of) the distance function $\delta_0$ assigning to each point the distance to the nearest boundary point. The "hilltops" of the heaps are thus represented by the local maximum points of $\delta_0$, which is at the same time the set of singularities of $\delta_0$. The number $S$ of these singularities obviously depends on the numbers $|E|$, $|V|$, and $|\partial\Gamma|$ of edges, vertices, and boundary points, respectively. As will be demonstrated below, it is in fact already determined by them and may be explicitly computed, and thus is a purely topological quantity - provided that the "multiplicity" of singularities at vertices is appropriately defined.

## 6.3 Distance function and singularities

As a matter of fact, for generic networks we have the simple formula

$$S = |E| - |V| + |\partial\Gamma|.$$

Even though it is not hard to show, this result is not trivial, as one can imagine many possible deformations of $\Gamma$, which do preserve $|E|$, $|V|$, and $|\partial\Gamma|$, but change the lengths of the edges.

As an effect of such a deformation, the singularities will move along the edges, possibly jumping from one edge to another at a vertex, whereas others may vanish, melt together, or be newly created. Hence, from a physical point of view our result is not obvious. However, it turns out that the problem can be transformed into a question of abstract graph theory, a setting where the proof becomes straightforward and elegant, while, on the other hand, revealing a close relation to the concept of cycle rank of disconnected graphs.

Our result can also be interpreted in the framework of computer algorithms, where the so-called single-source shortest-path problem (see for instance [22], pp. 580) of weighted graphs is of great importance for many different applications. The goal is to find the shortest distance from a given source node to each other vertex with respect to a real-valued weight attribute associated with each edge. Many algorithms have been designed to attack this problem, among which the probably most famous is known as *Dijkstra's algorithm* [23]. Later on we will embed this algorithm into our context, discuss the more general multiple-source shortest-path problem, and relate it to our result.

## 6.4 Singular points

As said before, we are concerned with the question of how many "hilltops" the maximal heaps will possess. However, the term "hilltop" is somewhat misleading at transition vertices, whence we prefer to speak of "singular points" which are defined by the property that there are more than one different directions leading to a shortest way to the boundary – a number which is generally smaller than the actual number of all different shortest ways from the singular point to the boundary. Let $s$ be such a singular point. If $s$ is not a transition vertex, it is clear that we have exactly two "shortest" directions. On the other hand, if $s$ coincides with a transition vertex, we might have more than two shortest directions to choose from, which should be appropriately taken into account. In fact, it is easy to see that if we introduce a suitable topology on the set of all networks, we may always slightly deform the network in such a way, that $s$ moves away from the transition vertex while splitting into several different singular points – exactly one less as there have been directions to choose a shortest way from. In this *generic* case our problem assumes its most appealing form, as singular points only occur on the edges. In order to conceive these ideas more precisely, we consider topological networks as special cases of *weighted graphs*, motivated by the following observation:

> *An edge $e \in E$ with end points $v, w \in V$ contains a singular point $s \notin V$ if and only if $|\delta_0(v) - \delta_0(w)| < l(e)$, where $l(e)$ is the length of the Jordan curve $e$. Furthermore, a transition vertex point $v \in V$ has exactly as many "shortest directions to the boundary" as it has "lower incident edges", that is, edges $e \in E$ for which there exists a vertex $w \in V$ such that $e = vw$ and $\delta_0(v) = \delta_0(w) + l(e)$.*

## 6.5 Weighted Graphs and Main Result

In this section we restrict our considerations to the abstract notion of weighted graphs and derive our main result within the framework of graph theory, whereby the proof becomes short and transparent. Note that whereas the physical problem – at least its technical realisation – requires a planar graph, all the following arguments equally apply if we admit any kind of connected graph $G$ with $V \neq \emptyset$; the only restriction we need is that $G$ be finite, that is, $|V| < \infty$. We even allow for multiple edges and loops, i.e. edges of the form $vv$, $v \in V$.

**Definition 6.1** *A* weighted graph $(G, l)$ *is a graph* $G = G(V, E)$ *equipped with a* weight function $l : V \to ]0, \infty[$.

**Definition 6.2** *Let $(G, l)$ be a weighted graph. For any path $P = v_1 \dots v_n$ of $G$ we define its length $l(P)$ by $l(P) := \sum_{i=1}^{n-1} l(v_i v_{i+1})$.*

It is then easy to verify the following

**Lemma 6.1** *Let $(G, l)$ be a connected weighted graph. Then the mapping $d : V \times V \to \mathbb{R}$ given by $d(v, w) := \min\{l(P) \,|\, P$ is a path connecting $v$ and $w\}$ is a metric on $V$.*

**Example.** Let $(G, l)$ be a weighted graph. Suppose also that $G$ is a connected topological graph, and assume that the weight function $l(e)$, $e \in E$, be given by the respective length of the Jordan curve $e$. Then $d$ is the restriction to $V \times V$ of the metric $d$ introduced earlier.

Again some vertices should be marked as "boundary" points, which motivates the following

**Definition 6.3** *A* weighted graph with boundary $(G, l, \partial G)$ *is a connected weighted graph* $(G, l)$ *with a (formal) boundary $\partial G$, where the* set of boundary vertices $\partial G \neq \emptyset$ *is a given subset of $V$. We call $V_T := V \backslash \partial G$ the* set of transition vertices.

Consequently, we introduce the corresponding distance function (which again is the restriction of the distance function introduced above, whenever $G$ is also a topological graph and $l$ is defined by the lengths of the edges).

**Definition 6.4** *Let $(G, l, \partial G)$ be a weighted graph with boundary. Then $\varphi : V \to \mathbb{R}$ given by $\varphi(v) := \min_{u \in \partial G} d(v, u)$ is called the* distance function *of $G$.*

The following definitions capture the idea mentioned in the preceding section of how to count singularities within the framework of weighted graphs.

**Definition 6.5** *Let $(G, l, \partial G)$ be a weighted graph with boundary. We call*

$$E_S := \{e = vw \in E \text{ with } l(e) > |\varphi(v) - \varphi(w)|\}$$

*the* set of singular edges of $G$, *whereas $E_R := E \backslash E_S$ is called the* set of regular edges of $G$.

**Definition 6.6** *Let $(G, l, \partial G)$ be a weighted graph with boundary. We define the* order of singularity of the edges $\kappa^E : E \to \mathbb{N}$ *by*

$$\kappa^E(e) := \begin{cases} 1 & \textit{if } e \in E_S \\ 0 & \textit{if } e \in E_R. \end{cases}$$

**Definition 6.7** *Let $(G, l, \partial G)$ be a weighted graph with boundary. We define the* order of singularity of the vertices $\kappa^V : V \to \mathbb{N}$ *by*

$$\kappa^V(v) := |\mathrm{Inc_L(v)}|,$$

*where* $\mathrm{Inc_L(v)} := \{\mathrm{vw} \in \mathrm{E}$ *with* $\varphi(v) = \varphi(w) + l(vw)\}$ *be the set of* lower incident edges of $v$ in $G$.

Note that we have $\kappa^V(v) = 0$ for each $v \in V_B$ and $\kappa^V(v) \geq 1$ for each $v \in V_T$. Furthermore, we can always achieve that we have $\kappa^V(v) = 1$ for all $v \in V_T$, if we arrange that all possible paths to the boundary in $G$ have pairwise different lengths by slightly varying the weight function $l$. We define

**Definition 6.8** *Let $(G, l, \partial G)$ be a weighted graph with boundary. $G$ is said to be* generic*, if the numbers $d(v, w)$ are pairwise different for all choices $v \in V_T$ and $w \in \partial G$.*

If $G$ is generic we clearly have $\kappa^V(v) = 1$ for each transition vertex $v \in V_T$. Furthermore, $v$ does then *not* carry a singularity and should therefore not contribute to the number of singularities of $G$. Hence, when counting the singularities in the general case, each transition vertex should contribute a number of singularities which equals its order of singularity minus one. Accordingly we define

**Definition 6.9** *Let $(G, l, \partial G)$ be a weighted graph with boundary. The order of singularity $S$ is then defined to be*

$$S := K - |V_T|,$$

*with*

$$K := \sum_{e \in E} \kappa^E(e) + \sum_{v \in V} \kappa^V(v).$$

By construction, the order of singularity $S$ of $G$ is exactly the quantity which counts the singularities of $G$ with the correct treatment of vertex points. This is also confirmed by the following

**Lemma 6.2** *Let $S$ be the order of singularity of a weighted graph with boundary $(G, l, \partial G)$ and let $\varepsilon > 0$. Then there is a weight function $l_\varepsilon$ with $\max_{e \in E} |l(e) - l_\varepsilon(e)| \leq \varepsilon$ such that $(G, l_\varepsilon, \partial G)$ is generic and such that the order of singularity $S_\varepsilon$ of $(G, l_\varepsilon, \partial G)$ satisfies $S_\varepsilon = S$ and equals the number of singular edges of $(G, l_\varepsilon, \partial G)$.*

**Proof.** If $(G, l, \partial G)$ is generic, there is nothing to show. If not, there is at least one vertex $v$ satisfying $|\mathrm{Inc_L(v)}| > 1$. Choose any edge $vw \in \mathrm{Inc_L(v)}$. We then increase $l(vw)$ by a value $\delta$ with $0 < \delta < \varepsilon$ chosen sufficiently small such that the values $\varphi(u)$ are not affected for all $u \in V$. This is possible as $v$ has at least one other lower adjacent vertex. Observe that as a consequence the order of singularity $\kappa^V(v)$ is decreased by one, as $vw$ will no longer be a lower incident edge. On the other hand, the value of $\kappa^E(vw)$ switches from 0 to 1, as $vw$ now satisfies $l(vw) > |\varphi(v) - \varphi(w)|$. Hence $S$ remains constant. We now repeat this procedure to finally eliminate all multiple lower incident edges while keeping $S$ constant, whereby, by construction, the length of each edge is at most increased by $\varepsilon$. $\square$

In fact, $S$ is completely independent of the choice of the weight function $l$ and may be explicitly computed, as it is stated in the following main theorem.

**Theorem 6.1** *Let $(G, l, \partial G)$ be a weighted graph with boundary. Then its order of singularity $S$ satisfies*

$$S = |E| - |V| + |\partial G|.$$

**Proof.** For any $v \in V$ and any $e = vw \in \mathrm{Inc_v}$ we introduce the quantity

$$\sigma_v(e) := \begin{cases} -1 & \text{if } \varphi(v) = \varphi(w) + l(vw) \\ 1 & \text{otherwise.} \end{cases}$$

Then we obtain the identities

$$\kappa^V(v) = \frac{1}{2} \sum_{e \in \mathrm{Inc_v}} (1 - \sigma_v(e)) \quad \text{and} \quad \kappa^E(e) = \frac{1}{2}(\sigma_v(e) + \sigma_w(e))$$

for $e = vw$. Using this, we compute by means of definition 6.9

$$K = \frac{1}{2} \left[ \sum_{v \in V} \sum_{e \in \mathrm{Inc_v}} (1 - \sigma_v(e)) + \sum_{e = vw \in E} (\sigma_v(e) + \sigma_w(e)) \right] = \frac{1}{2} \sum_{v \in V} \deg v = |E|.$$

The assertion then follows by definition 6.9 and by the relation $|V_T| = |V| - |\partial G|$. $\square$

**Corollary 6.1** *Let $(G, l, \partial G)$ be a generic weighted graph with boundary. Then the number $|E_S|$ of singular edges may be computed by*

$$|E_S| = |E| - |V| + |\partial G|.$$

**Proof.** As $G$ is generic, we have $S = |X_S|$. $\square$

## 6.6 Non-Uniqueness of Shortest Path Algorithms

As graphs are persuasively simple and versatile objects, graph theoretical problems subsume many concrete problems arising in the theory of computer algorithms. A prominent question is how to efficiently detect shortest paths connecting a given vertex with prescribed *source* vertices in a weighted graph. Dijkstra's classical algorithm [23] basically was the first managing the situation where there is exactly one source vertex (single-source shortest path problem), and is followed by a wide list of modifications or more specific approaches to the same problem (see for instance [24], [25], [26], and [27]).

Dijkstra's algorithm successively lists all shortest paths from the single source to the other vertices, starting at the source vertex. In terms of a topological graph $G$ equipped with a metric $d$ and its associated topological network $\Gamma$ as introduced in section 4.3 it does the following: Starting at the source vertex $s$ it determines the level sets $L_t := \{x \in \Gamma \,|\, \delta_0(x) = t\}$ of the distance function $\delta_0(x) := d(x, s), x \in \Gamma$, for continuously increasing $t \geq 0$. As soon as a set $L_t$ contains one or more vertices, these vertices are assigned the shortest distance $t$ to the source, along with the way "back downhill" as shortest path to the source (which is not necessarily unique). This procedure is continued until a shortest path is assigned to each vertex. Whereas Dijkstra's algorithm is originally restricted to weighted graphs (that is, detection of shortest ways between vertices), the level set idea described here may of course be extended to all points $p \in \Gamma$, particularly to edge points.

The relation to the concept of the previous section is now easily established: Let $(G, l, \partial G)$ be a weighted graph with boundary, and assume $G$ also to be a topological graph such that $l$ coincides with the edge lengths. If $G$ has only one boundary vertex $s$, it may also be considered as a weighted graph with a single source. Then every singular edge of $G$ contains exactly one point $p$ with the property, that the level set extension of Dijkstra's algorithm yields at least two shortest paths to $s$ *which do not coincide near* $p$. Similarly, a singular vertex $v$ is assigned shortest paths leaving it in exactly $\kappa^V(v) - 1$ different directions. In other words, here the single-source shortest path problem cannot be solved uniquely in the sense, that the direction in which shortest ways lead be unique.

Of course one can also consider multiple-source shortest path problems, represented by weighted graphs with more than one boundary vertex. Again, the singular points are exactly those points for which this problem cannot be solved uniquely in the above sense. Thus, theorem 6.1 captures and quantifies the "non-uniqueness" of such shortest path problems.

## 6.7 Singular Points and Cycle Rank

It is worth mentioning the close relation of our result to the concept of cycle rank of abstract graphs. Let $G = (V, E)$ with $V = \{v_1, \dots, v_p\}$ and $E = \{e_1, \dots, x_q\}$ be a graph. A *cycle* in $G$ is a path whose endpoints coincide. Analogously to techniques in algebraic topology a 0-*chain* of $G$ is a formal linear combination $\sum \varepsilon_i v_i$, where $\varepsilon_i \in F_2$ (the field with two elements), whereas 1-*chains* are formal sums $\sum \varepsilon_j e_j$ of edges. The *boundary operator* $\partial$ sends 1-chains to 0-chains in such a way that $\partial$ is linear and that we have $\partial e = v + w$ for edges of the form $e = vw$. A 1-chain with boundary 0 is called *cycle vector* and can be regarded as a set of edge-disjoint cycles. The collection of all cycle vectors forms a vector space over $F_2$ called the *cycle space* of $G$. A *cycle basis* is defined as a basis for the cycle space of $G$ consisting entirely of cycles. A cycle vector $Z$ is said to be dependent on the cycles $Z_1, \ldots, Z_k$, if it can be written as $\sum_{i=1}^k \varepsilon_i Z_i$. Thus a cycle basis of $G$ is a maximal collection of independent cycles of $G$, or a minimal collection of cycles on which all cycles depend. The number $m(G)$ of cycles in a cycle basis of $G$ is called *cycle rank*, and it can be shown (see for instance [18], pp. 37-39) that we have

$$m(G) = q - p + k,$$

where $k$ is the number of *components* of $G$, that is, the maximal number of subsets of $V$ with the property that any two elements of two different subsets are not connected by a path.

The connection of this concept to our problem is established as follows. We first recall

**Definition 6.10** *A* subgraph *$H = (V_H, E_H)$ of the graph $G = (V_G, E_G)$ is a graph with $V_H \subset V_G$ and $E_H \subset E_G$. A subgraph $H$ of $G$ is said to be* spanned *by $V_H \subset V_G$, if $vw \in E_H$ for each edge $vw \in E_G$ with $v, w \in V_H$.*

According to lemma 6.2 we can always make any weighted graph generic by a small perturbation of the length function without $S$ being changed. We therefore restrict our considerations to generic graphs. Let $(G, l, \partial G)$ be a generic weighted graph with boundary. The genericity ensures that singularities only occur on edges. We then "disconnect" $G$ by deleting certain edges in the following way: For each boundary point $u \in \partial G$ we form the subgraph $G_u$ spanned by

$$V_u := \{v \in V \mid d(u, v) = \min_{w \in \partial G} d(w, v)\}.$$

Observe that the sets $V_u$ form a partition of $V$, i.e. they are pairwise disjoint and $\dot\cup_{u \in \partial G} V_u = V$. The union of the graphs $G_u, u \in \partial G$, forms the graph $\tilde{G}$ which is obtained by deleting the set $E_0$ of exactly those edges from $G$, whose endpoints are not contained in the same vertex set $V_u$. Observe that the deleted edges are all singular, which is easy to see by the definition of the subgraphs $G_u$. The remaining graph $\tilde{G}$ has exactly $|\partial G|$ components. According to the formula for the cycle rank, its cycle rank $m = m(\tilde{G})$ satisfies $m = |E \backslash E_0| - |V| + |\partial G|$.

Now consider $G_u$ for any fixed $u \in \partial G$. As $G_u$ has only one component, the cycle rank $m_u$ of $G_u$ is $m_u = |E_u| - |V_u| + 1$, where $E_u$ is the edge set of $G_u$. Clearly we have $m = \sum_{u \in \partial G} m_u$. We now claim that $E_u$ contains exactly $m_u$ singular edges – in other words, each element of the cycle space basis of $G_u$ contributes one singular edge. The argument is the following: It is clear that whenever $G_u$ has at least one cycle, it must also contain at least one singular edge. Deleting all singular edges from $G_u$ thus leaves a *tree*, that is, a cycle-free graph. Successively re-adding the deleted singular edges then yields one new independent cycle per singular edge. Any other cycle or cycle vector depends on these cycles; the maximal number of independent cycles – the cycle rank $m_u$ – thus exactly equals the number of singular edges in $G_u$.

Altogether we obtain for the order of singularity of $G$:

$$S = |E_0| + \sum_{u \in \partial G} m_u = |E_0| + m = |E_0| + |E \backslash E_0| - |V| + |\partial G| = |E| - |V| + |\partial G|.$$

We thus have recovered corollary 6.1.

# Chapter 7

# Viscosity solutions on $n$-dimensional ramified spaces

**Summary.** The theory of the present chapter generalizes the results obtained in chapter 5. We generally introduce higher dimensional ramified manifolds and, as special cases, locally elementary polygonal ramified spaces (LEP spaces, for short). To the latter we generalize the theory of viscosity solutions of Hamilton-Jacobi equations presented in chapter 5.

## 7.1 Introduction

In the literature there are many different ways of introducing "ramified spaces" (cs. Lumer, Nicaise, Ali Mehmeti) or "branched manifolds" (cs. Williams). The definitions vary in different aspects, depending on the kind of theory to be developed. In a general approach, subsets of classic differentiable manifolds are glued together at parts of their boundaries by means of the topological gluing operation. Another, more specific definition, demands the unicity of the "tangent space" at ramification points (cf. Williams) by describing how the branches should be situated relatively to each other in the ambient space. For our purposes we choose a rather intuitive approach, which is very similar to the concept of a manifold with boundary. The basic idea is that in contrast to classic topological manifold, a *ramified* topological manifold should not only contain points around which it is locally homeomorphic to Euclidean space (*simple points*), but should also allow for *ramification points* around which it is locally homeomorphic to some kind of *ramified* Euclidean space. The latter is visualized as a collection of closed Euclidean half spaces glued together at their boundary hyperplanes and will be called *elementary ramified space*. Small neighborhoods of a given ramification point are then separated into different *branches* corresponding to the branches of the homeomorphic elementary ramified space. We endow these ramified topological manifolds with suitable differentiable structures, leading to an extension of the concept of tangent space at ramification points. This generalization has to take account of the idea that a real function defined in a neighborhood of such a point can be differentiated in direction of all different branches. Put in other terms, each branch "contributes" a different tangent space. In particular, we dispense with the uniqueness of the tangent space at ramification points. Once we have introduced the differentiable structure, we will see that the set $\Sigma$ of all ramification points forms a subset which itself is a differentiable manifold in the classic sense. Hence, for each of the branches contacting at a ramification point $x \in \Sigma$, one can speak of the *normal direction* at $x$ on this branch with respect to $\Sigma$. The possibility to differentiate into normal directions at ramification points is crucial for our theory, as it will turn out that a general definition of viscosity solutions on ramified manifolds requires this very possibility. The reason for this requirement is the fact that at a ramification point the transition condition for viscosity solutions will make use of certain test functions whose normal derivatives coincide in modulus but differ in sign at the point in question. This is the reason why we unfortunately will not be able to extend the theory to more complicated ramified objects such as the surface of a cube, where at corner points one obviously has no normal derivatives at one's disposal. As a matter of fact, in order not to make things too general we chose to restrict ourselves to a rather simple kind of ramified manifolds, the so-called *local elementary polygonal ramified spaces (LEP spaces)*. LEP spaces are ramified spaces according to Lumer such that each branch is a flat $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. At the same time, we demand that they are ramified manifolds in the above sense. Hence they basically can be visualized as polygonal subsets of hyperplanes in $\mathbb{R}^{n+1}$ which are glued together at certain edges, with the restriction that "corner points" cannot occur. Also, the respective sets of ramification points with a fixed number of branches are of pairwise positive distance to each other. The term "local elementary" refers to the fact described above: that they either are locally homeomorphic to Euclidean space or to elementary ramified space. As we demand the elementary ramified spaces to have at least two branches, LEP spaces consequently do not contain boundary points around which they locally look like Euclidean half-space. However, as we discuss boundary value problems, we will require them to have a suitable relative boundary.

An extension of the vanishing viscosity method of chapter 4 to LEP spaces is not straightforward. This, however, is not due to the problem of convergence of the 1-parameter family of "viscous" solutions of a given Hamilton-Jacobi boundary value problem. The difficulty rather arises from the mere question of *existence* of this family. Grisvard has shown that already a theory of linear boundary value problems on polygonals brings up severe problems regarding the existence of regular bounded solutions. Also, to provide an existence theory for quasilinear boundary value problems on LEP spaces with transition conditions would go beyond the scope of this work. Instead we focus on the convergence problem and show that such a family of viscous solutions will converge, *whenever it exists.*

## 7.2 Ramified manifolds and LEP spaces

As mentioned above, we want to define ramified manifolds as objects which locally resemble either (non-ramified) Euclidean space or some ramified parameter space. We begin with the introduction of the latter, the *elementary ramified spaces (of order r).*

**Definition 7.1** *Let $n \geq 1$. We respectively define the $n$-dimensional open and closed Euclidean half-space by*

$$\mathbb{R}^n_{>0} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 > 0\} \quad \textit{and} \quad \mathbb{R}^n_{\geq 0} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, x_1 \geq 0\}.$$

**Definition 7.2** *Let $n \geq 1$ and let $r \geq 2$ be an integer. Set*

$$\tilde{\mathcal{R}}^n_r := \mathbb{R}^n_{\geq 0} \times \{1, \ldots, r\} \quad \textit{and} \quad \mathcal{R}^n_r := \tilde{\mathcal{R}}^n_r / R,$$

*where $R$ be the equivalence relation on $\tilde{\mathcal{R}}^n_r$ which for each choice of $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ identifies all points $((0, x_2, \ldots, x_n), j) \in \tilde{\mathcal{R}}^n_r$, $1 \leq j \leq r$. Equivalence classes with respect to $R$ are denoted by an upper bar.*

*Let $\tilde{\mathcal{R}}^n_r$ carry the product topology formed by the Euclidean topology on $\mathbb{R}^n_+$ and by the discrete topology on $\{1, \ldots, r\}$. The quotient topology of this product topology with respect to the quotient mapping induced by $R$ turns $\mathcal{R}^n_r$ into a topological space which we denote by the $n$-*dimensional elementary ramified space of order $r$.

*Furthermore define the corresponding* ramification space

$$\Sigma^n_r := \{\overline{((0, x_2, \ldots, x_n), j)} \mid (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}, 1 \leq j \leq r\} \subset \mathcal{R}^n_r$$

*and the open and closed* branches

$$\mathcal{R}^n_{r,j} := \{\overline{(x, j)} \in \mathcal{R}^n_r, x \in \mathbb{R}^n_{>0}\} \quad \textit{and} \quad \bar{\mathcal{R}}^n_{r,j} := \{\overline{(x, j)} \in \mathcal{R}^n_r, x \in \mathbb{R}^n_{\geq 0}\}$$

**Remark.** Note that for $n \geq 2$ we can identify $\mathcal{R}^n_r$ with $\mathcal{R}^1_r \times \mathbb{R}^{n-1}$, $\bar{\mathcal{R}}^n_r$ with $\bar{\mathcal{R}}^1_r \times \mathbb{R}^{n-1}$, as well as $\Sigma^n_r$ with $\Sigma^1_r \times \mathbb{R}^{n-1} = \{0\} \times \mathbb{R}^{n-1}$, where $0 = \overline{(0, j)}$. Furthermore observe that $\Omega := \mathcal{R}^n_r$ is a ramified space according to Lumer's definition with $\Omega_j = \mathcal{R}^n_{r,j}$, $1 \leq j \leq r$, and $N_r = \Sigma^n_r$.

We obviously can interpret $\mathcal{R}^n_r$ as the subset of $\mathbb{R}^{n+1}$ given by

$$\Big(\cup_{j=0}^{r-1} \{(r \cos(2j/(r\pi)), r \sin(2j/(r\pi))) \in \mathbb{R}^2, r \geq 0\}\Big) \times \mathbb{R}^{n-1},$$

and it is easy to verify that the subset topology on this subset coincides with the topology on $\mathcal{R}^n_k$ given by the definition above. Both topologies in turn coincide with the topology on $\mathcal{R}^n_k$ induced by the path metric $\operatorname{dist}(\cdot, \cdot)$. Having this in mind, we provide a fundamental lemma about homeomorphisms from $\mathcal{R}^n_k$ to itself.

**Proposition 7.1** *Let $n \geq 1$ and $r \geq 3$. Let $x \in \Sigma_r^n$ and let $U \subset \mathcal{R}_r^n$ be an open connected set with $x \in U$. Furthermore let $\varphi : U \to \mathcal{R}_s^n$ be a homeomorphism.*

*(i) We have $\varphi(x) \in \Sigma_s^n$.*

*(ii) We have $s = r$.*

*(iii) For each $1 \leq j \leq r$ there is a $1 \leq k \leq r$ such that $\varphi(U \cap \mathcal{R}_{r,j}^n) \subset \mathcal{R}_{r,k}^n$.*

**Definition 7.3** *Let $x \in \mathcal{R}_k^n$ and $\varepsilon > 0$. The open ball around $x$ with radius $\varepsilon$ is defined by $B_\varepsilon(x) := \{y \in \mathcal{R}_k^n \mid dist(y, x) < \varepsilon\}$.*

**Definition 7.4** *Let $m, n \geq 1$, $r \geq 2$, let $U \subseteq \mathcal{R}_r^n$ be a domain, and let $f : U \to \mathbb{R}^m$ be continuous. Then $f$ is said to be $C^l$-differentiable at $x \in U$ $(1 \leq l \leq \infty)$, if the following holds.*

*(i) If $x \in \Sigma_n^k$, then for each $1 \leq j \leq r$ there is a domain $V_j \subset \mathbb{R}^n$ and an $l$ times continuously differentiable function $f_j : V_j \to \mathbb{R}^m$ such that, if $\bar{\mathcal{R}}_{r,j}^n$ is interpreted as the closed half-space $\mathbb{R}_{\geq 0}^n$, we have $x \in V_j$ and $f_j \equiv f$ on $V_j \cap \mathbb{R}_{\geq 0}^n$.*

*(ii) If $x \in \mathcal{R}_{r,j}^n \simeq \mathbb{R}_{>0}^n$ for some $1 \leq j \leq r$, then $f$ is $l$ times continuously differentiable at $x$ in the classical sense.*

**Definition 7.5** *Let $r \geq 3$ and let $U \subset \mathcal{R}_r^n$ be a domain with $U \cap \Sigma_r^n \neq \emptyset$. A homeomorphism $\varphi : \mathcal{R}_r^n \supset U \to \mathcal{R}_r^n$ is called* diffeomorphism *if the respective restrictions of $\varphi$ and $\varphi^{-1}$ to $\mathcal{R}_{r,j}^n \cap U$ and $\mathcal{R}_{r,j}^n \cap \varphi(U)$, are $C^\infty$-differentiable in the sense of definition 7.4 (note that the images of these restrictions can be thought of as subsets of $\mathbb{R}_{>0}^n$ according to proposition 7.1).*

We are now ready to define topological and differentiable ramified manifolds.

**Definition 7.6** *A set $M$ is called an $n$-*dimensional topological ramified manifold *if it is endowed with a Hausdorff topology and if for each point $x \in M$ there is an open set $U \subset M$ with $x \in U$ such that there is an integer $r = r(x) \geq 2$ and a homeomorphism $\boldsymbol{x} : U \to \mathcal{R}_r^n$ with $\boldsymbol{x}(x) \in \Sigma_r^n$. The number $r(x)$ is called* ramification order of $x$ *and is independent of $\boldsymbol{x}$ according to proposition 7.1.*

*A point $x \in M$ is called* simple point *if $r(x) = 2$ and* branching point *if $r(x) \geq 3$. The set of all branching points is denoted by $\Sigma(M)$ and is called* ramification space of $M$.

**Definition 7.7** *Let $M$ be an $n$-dimensional topological ramified manifold with ramification space $\Sigma$. We call $M$* differentiable *if the following is the case.*

*(i) There is a maximal family $(U_\alpha, \boldsymbol{x}_\alpha)$ of charts, i.e. of open sets $U_\alpha \subset M$ with $U_\alpha \cap \Sigma \neq \emptyset$ and homeomorphisms $\boldsymbol{x}_\alpha : U_\alpha \to \mathcal{R}_{r(\alpha)}^n$ with the following property.*

*If $V := U_\alpha \cap U_\beta \neq \emptyset$, the mapping $\varphi : V \to \mathcal{R}_r^n$ given by $\varphi := \boldsymbol{x}_\beta \circ \boldsymbol{x}_\alpha^{-1}$ is a diffeomorphism in the sense of definition 7.5. (Note that we have $r = r(\alpha) = r(\beta)$ according to proposition 7.1.)*

*(ii)* $M \backslash \Sigma$ *is an* $n$*-dimensional differentiable manifold of dimension* $n$ *allowing for an atlas (i.e., a maximal family of charts)* $(V_\beta, \boldsymbol{y}_\beta)$ *which is consistent with* $(U_\alpha, \boldsymbol{x}_\alpha)$ *in the sense that for each pair* $\alpha, \beta$ *with* $W := U_\alpha \cap V_\beta \neq \emptyset$ *the chart* $(W, \boldsymbol{x}_\alpha|_W)$ *is contained in this atlas.*

*(iii)* $(\cup_\alpha U_\alpha) \cup (\cup_\beta V_\beta) = M$.

**Example.** Let $\Gamma$ be a topological network and $G = (V, E)$ the corresponding graph. Set $\tilde{\Gamma} := \Gamma \backslash \{v \in V \mid \deg(v) = 1\}$. Then for $n \geq 2$ the set $M := \tilde{\Gamma} \times \mathbb{R}^{n-1}$ is an $n$-dimensional topological ramified manifold. The set $M_r$ of points of ramification order $r$ is given by

$$M_r = \begin{cases} \cup_{\{v \in V \mid \deg(v) = r\}} (\{v\} \times \mathbb{R}^{n-1}) & \text{if } r > 2 \\ M \backslash \Sigma(M) & \text{if } r = 2. \end{cases}$$

Here we have $\Sigma(M) = \cup_{\{r>2\}} M_r$ .We call $M$ an $n$-dimensional network (cf. Nicaise).

Next we have to extend the notion of tangent space to differentiable ramified manifolds. In fact, the idea of interpreting tangent vectors as equivalence classes of curves in $M$ having the same derivative when composed with $C^\infty$-differentiable functions on $M$ can easily be transferred to ramification points.

**Definition 7.8** *Let* $M$ *be a differentiable ramified manifold and let* $1 \leq l \leq \infty$*. A continuous function* $f : M \to \mathbb{R}$ *is said to be* $C^l$-differentiable at $x \in M$, *if for any chart* $(U, \boldsymbol{x})$ *around* $x$ *the function* $f \circ \boldsymbol{x}^{-1}$ *is* $C^l$*-differentiable at* $x$ *according to definition 7.4.* $f$ *is called* $C^l$-differentiable in $V \subset M$, *if it is* $C^l$*-differentiable at all* $x \in V$.

**Remark.** In order not to cause any indexation problems, let us from now on assume without restriction of generality that for any two charts $(U, \mathbf{x})$ and $(V, \mathbf{y})$ around a given point $x \in \Sigma$ the mapping $\mathbf{y} \circ \mathbf{x}^{-1} : \mathcal{R}^n_{r(x)} \to \mathcal{R}^n_{r(x)}$ maps $\mathcal{R}_{r(x),j} \to \mathcal{R}_{r(x),j}$ for any $1 \leq j \leq r(x)$. Observe that the following definition will then be independent of the choice of the respective chart.

In the sequel let $M$ always be an $n$-dimensional differentiable ramified manifold with ramification space $\Sigma$. Let $x \in \Sigma$ and let $r := r(x)$ be the ramification order of $x$.

**Definition 7.9** *Let* $(U, \boldsymbol{x})$ *be a chart around* $x$*. We say that a continuous curve* $\alpha : (-\varepsilon, \varepsilon) \to M$ reaches $x$ from a proper branch, *whenever*

*(i)* $\alpha(0) = x$,

*(ii)* $\exists\, 1 \leq j \leq r$ *and a chart* $(U, \boldsymbol{x})$ *of* $x$ *with* $\tilde{\alpha}(t) := (\boldsymbol{x} \circ \alpha)(t) \in \mathcal{R}^n_{r,j}\ \forall\ t \in (-\varepsilon, 0)$.

*(iii)* $\tilde{\alpha}(t) := (\boldsymbol{x} \circ \alpha)(t) \in \mathcal{R}^n_{r,j}$

*In particular, we say that* $\alpha$ reaches $x$ from the branch $j$, *where the index* $j$ *is independent of the choice of* $U$ *in view of the above remark.*

**Remark.** Note that if (ii) holds for some chart around $x$, then it automatically holds for all charts around $x$ in view of proposition 7.1. Furthermore, when (ii) holds, then $\tilde{\alpha}$ maps $(-\varepsilon, 0)$ to $\mathcal{R}^n_{r,j} \simeq \mathbb{R}^n_+$ and (iii) makes sense.

We now subdivide the class of all curves reaching $x$ from a proper branch into equivalence classes, which then represent the tangent vectors.

**Definition 7.10** *Let $\mathcal{C}(x)$ be the set of all curves reaching $x$ from a proper branch and let $\alpha, \beta \in \mathcal{C}(x)$. We say that $\alpha$ and $\beta$ are equivalent, if for all functions $f : M \to \mathbb{R}$ which are $C^\infty$-differentiable at $x$ we have*

$$(f \circ \alpha)'(0) = (f \circ \beta)'(0),$$

*where both derivatives are left-sided. We then denote the set of all equivalence classes by the* tangent space $T_xM$ of $M$ at $x$. *We say that $\xi \in T_xM$ is a* $j$-ramified tangent vector at $x$, *if $\xi$ contains a curve reaching $x$ from the branch index $j$. We set $\xi(f) := (f \circ \alpha)'(0)$, $\alpha \in \xi$. The set of all $j$-ramified vectors at $x$ is denoted by $T^j_xM$, and each $\xi \in T_x\Sigma := \cap_j T^j_xM$ is called a* $j$-ramified tangent vector at $x$.

**Remark.** Clearly, $T_xM$ is no vector space. In fact, it can be identified with the elementary ramified space $\mathcal{R}^n_r$, where

$$T^j_xM \equiv \bar{\mathcal{R}}^n_{r,j},\ 1 \le j \le r, \text{ and } T_x\Sigma := \Sigma^n_r.$$

Hence $T_x\Sigma \subset T^j_xM$ can be identified with $\mathbb{R}^{n-1}$, whereas $T^j_xM$ can be identified with $\mathbb{R}^n_{\ge 0}$. For the purpose of shorter presentation, let us for each $1 \le j \le r$ agree to extend $T^j_xM$ to the entire $\mathbb{R}^n$ via the relation $\xi(f) = -\xi(-f)$ for any $f \in C^\infty(M)$ and any $\xi \in T^j_xM$. The advantage of this extension is that each $j$-tangent space at $x$ carries a vector space structure.

The following definitions will be required later on.

**Definition 7.11** *Let $1 \le j \le r$. Let $f : M \to \mathbb{R}$ be continuously differentiable at $x$ and let $\xi_1, \ldots, \xi_n$ be a basis of $T^j_xM$. We define the* $j$-gradient *by*

$$D^j f(x) := \sum_{i=1}^n \xi_i(f)\xi_i^\star \in (T^j_xM)^\star.$$

*Here $(T^j_xM)^\star$ is the dual space of $T^j_xM$, and $\xi_1^\star, \ldots, \xi_n^\star$ is the corresponding dual basis.*

**Definition 7.12** *Let $(U, \boldsymbol{x})$ be a chart around $x$ and let $1 \le j \le r$. By*

$$D^j \boldsymbol{x} : T^j_xM \to \mathbb{R}^n$$

*we denote the vector space isomorphism with the property*

$$\frac{\partial}{\partial v}(f \circ \boldsymbol{x}^{-1}) = \xi(f),$$

*where* $v := D^j \boldsymbol{x}(\xi)$.

Having introduced the general concept of ramified manifolds, we now have the framework available to generalize the theory of chapter 5. However, our opinion is that a generic theory of viscosity solutions on ramified manifolds is a rather straightforward extension of the specific theory we decided to present here. In fact, the development of a theory of viscosity solutions on general Riemannian manifolds is not within the scope of this work. We are rather interested in the phenomena occurring at the ramification space and have therefore decided to only consider flat ramified manifolds in the following sense.

**Definition 7.13** *Let* $\Omega = (\Omega^\star, \mathcal{L}, N_r)$ *be a ramified space according to definition 4.1. We call* $\Omega$ *an* ($n$-dimensional) polygonal ramified space *if the following conditions are satisfied.*

*(i) For each* $j \in J$ *there is a hyperplane* $P_j$ *of* $\mathbb{R}^{n+1}$ *such that* $\Omega_j \subset P_j$.

*(ii)* $\Omega^\star = \cup_{j\in J}\bar{\Omega}_j$, *where the closure is taken with respect to the subset topology of* $P_j$ *induced by* $\mathbb{R}^{n+1}$.

*(iii)* $\Omega^\star$ *carries the subset topology induced by* $\mathbb{R}^{n+1}$.

*For each* $j \in J$ *we set*

$$\partial_e\Omega_j := \partial\Omega_j \cap N_e^\star \quad \textit{and} \quad \partial_r\Omega_j := \partial\Omega_j \cap N_r.$$

*The set* $N_e^\star$ *is also called* boundary $\partial\Omega$ of $\Omega$.

Note that the notion of polygonal ramified space does not imply that it is locally homeomorphic to an elementary ramified space around ramification points. As we do require this property, we strengthen the concept in the following definition.

**Definition 7.14** *An* $n$*-dimensional polygonal ramified space* $\Omega \subset \mathbb{R}^{n+1}$ *is called* locally elementary, *if it is a differentiable ramified manifold with* $\Sigma(\Omega) = N_r$. *We call locally elementary polygonal ramified spaces* LEP spaces, *for short.*

**Example.** Any $n$-dimensional network as defined above is an LEP space.

**Definition 7.15** *For each* $j \in J$ *we set*

$$\tilde{\Omega}_j := \Omega_j \cup \partial_r\Omega_j.$$

*For any* $x \in \Sigma$ *we set* $\mathrm{Inc_x} := \{\mathrm{j} \in \mathrm{J} \,|\, \mathrm{x} \in \partial\Omega_\mathrm{j}\}$.

In the sequel let $\Omega$ always denote an $n$-dimensional LEP space, $n \geq 2$, with finitely many branches $\Omega_j$, $j \in J$. Furthermore let $\Sigma := \Sigma(\Omega) = N_r$ denote the ramification space of $\Omega$.

LEP spaces have the convenient property that around any given point we can always choose a chart induced by the canonical identification with the Euclidean or a suitable elementary ramified space. Although this property is obvious, we state the following proposition.

**Proposition 7.2** *For any $x \in \Omega$ there is a neighborhood $V_x$ of $x$ and a canonical identification*

$$\boldsymbol{i}_x : V \to \boldsymbol{i}_x(V),$$

*where $\boldsymbol{i}_x(V) \subset \mathbb{R}^n$, $\boldsymbol{i}_x(x) = 0$, if $x \notin \Sigma$, and $\boldsymbol{i}_x(V) \subset \mathcal{R}^n_{r(x)}$, $\boldsymbol{i}_x(x) = 0$, if $x \in \Sigma$. In the latter case, $\boldsymbol{i}_x$ induces a bijective mapping $\mathcal{I}_x$ between the index set* $\mathrm{Inc_x}$ *and the set $\{1, \ldots, n\}$.*

### 7.2.1 Hamilton-Jacobi equations on LEP spaces

We now want to consider first order Hamilton-Jacobi equations and corresponding boundary value problems on LEP spaces.

**Definition 7.16** *A* Hamiltonian *$H = (H^j)_{j \in J}$ is a family of mappings $H^j$, such that for fixed $j \in J$ the mapping $H^j$ assigns to each $x \in \tilde{\Omega}_j$ a function*

$$H^j(\cdot, \cdot, x) : T_x\tilde{\Omega}_j \times \mathbb{R} \to \mathbb{R},$$

*where we set $T_x\tilde{\Omega}_j := T^j_x\Omega$ if $x \in \Sigma$. We furthermore assume the mappings $H^j$ to be twice continuously differentiable in the following sense: For fixed $x \in \tilde{\Omega}_j$ the above identification allows us to think of $H^j$ as a mapping*

$$H^j : \mathbb{R}^n \times \mathbb{R} \times V \to \mathbb{R}$$

*where $V$ is an open neighborhood of $0 \in \mathbb{R}^n$ or $0 \in \mathbb{R}^n_{\geq 0}$, if $x \in \Omega_j$ or $x \in \Sigma$, respectively, and we assume these mappings to be $C^2$ in each argument. In the latter case we mean $C^2$ in the sense of definition 7.8.*

Let us assume from now on that $H$ be a Hamiltonian such that each $H^j$, $j \in J$, has the following properties under the canonical identification at any fixed $x \in \Omega$:

$$\begin{array}{ll} (i) & H^j(0,z,0) < 0 \text{ for all } z \in \mathbb{R} \\ (ii) & H^j(p,\cdot,0) \text{ is non-decreasing for all } p \in \mathbb{R}^n \\ (iii) & H^j(p,z,0) \to \infty \text{ as } |p| \to \infty \text{ for all } (p,z) \in \mathbb{R}^n \times \mathbb{R} \\ (iv) & H^j(p,z,0) \text{ is convex in } p \text{ for all fixed } z \in \mathbb{R} \end{array} \tag{7.1}$$

Furthermore we need the $H^j$ to be continuous across the ramification space, as well as rotationally symmetric in $p$. In fact, after the canonical identification we demand at each $x \in \Sigma$

$$\begin{array}{ll} (v) & H^j(p,z,0) = H^k(p,z,0) \text{ for all } z \in \mathbb{R},\, p \in \mathbb{R}^n \text{ and all } j,k \in \mathrm{Inc_x} \\ (vi) & H^j(p,z,0) = H^j(\tilde{p},z,0) \text{ for all } p,\tilde{p} \in \mathbb{R}^n \text{ with } |\tilde{p}| = |p|, \text{ and all } z \in \mathbb{R},\, j \in \mathrm{Inc_x}. \end{array} \tag{7.2}$$

Moreover we assume

$$|D^j H^j(p,z,x)| \le C(x,z) \quad \text{for all } p \in (T_x\Omega_i)^\star \text{ and all } z \in \mathbb{R},\, x \in \Omega_j,\, j \in J. \tag{7.3}$$

**Test functions**

Our aim is to transfer the test function technique employed in chapter 5 to LEP spaces. For this purpose we extend the necessary concepts, such as differentiability across the ramification space. From now on we will for each $x \in \Omega$ fix a canonical identification chart $(V_x, \mathbf{i}_x)$ as defined in proposition 7.2. We note, however, that all concepts to be discussed do not depend on the choice of this chart.

**Definition 7.17** *Let $r > 3$, $n \ge 2$, and let $x \in \Sigma^n_r \subset \mathcal{R}^n_r$. Furthermore let $u : \mathcal{R}^n_r \to \mathbb{R}$ be a function which is continuously differentiable at $x$. We then denote by $\partial_1 u(x), \dots, \partial_{n-1}(x)$ the directional derivatives of $u$ at $x$ with respect to the canonical basis vectors $e_1, \dots, e_{n-1}$ of $\Sigma^n_{r(x)} \equiv \mathbb{R}^{n-1}$. For $1 \le j \le r$ we furthermore denote by $\partial_{\nu_j} u(x)$ the directional derivative of $u$ at $x$ with respect to the inward pointing unit normal $\nu_i$ of $\mathcal{R}^n_{r,j} \equiv \mathbb{R}^n_{\ge 0}$.*

**Definition 7.18** *Let $x \in \Sigma$, $r := r(x)$. Furthermore let $u : V \to \mathbb{R}$ be a function which is continuously differentiable at $x$. We set*

$$\partial_i u(x) := \partial_i (u \circ \boldsymbol{i}_x^{-1})(0),\ 1 \le i \le n-1 \quad \textit{and,} \quad \partial_{\nu_j} u(x) := \partial_{\nu_{\mathcal{I}_x(j)}} (u \circ \boldsymbol{i}_x^{-1})(0),\ j \in \mathrm{Inc_x}.$$

*Here $\mathcal{I}_x$ is the relabel map defined in proposition 7.2.*

**Definition 7.19** *Let $u : \Omega \to \mathbb{R}$. We denote its restriction to $\Omega_j$ by $u^j$, $j \in J$. Furthermore, let $C(\Omega)$ be the space of all continuous functions on $\Omega$.*

The natural generalization of $k,l$-differentiability introduced in chapter 5 is given by the following definition.

**Definition 7.20** *Let $x \in \Sigma$, $\varphi \in C(\Omega)$, and $k,l \in \mathrm{Inc_x}$, $k \neq l$. We call $\varphi$ $k,l$*-differentiable at *$x$, if $\varphi$ is continuously differentiable at $x$, and if we have*

$$\partial_{\nu_k} \varphi(x) + \partial_{\nu_l} \varphi(x) = 0.$$

**Definition 7.21** *Let $u \in C(\Omega)$ and let $\varphi \in C(\Omega)$.*

*Let $j \in J$ and let $x \in \Omega_j$. We call $\varphi$ an* upper (lower) test function of $u$ at $x$, *if $u - \varphi$ attains a local maximum (minimum) at $x$ and if $\varphi$ is continuously differentiable at $x$.*

*Let $x \in \Sigma$ and let $k, l \in \mathrm{Inc_x}$, $k \neq l$. We call $\varphi$ an* upper (lower) $k, l$-test function of $u$ at $x$, *if $u - \varphi$ attains a local maximum (minimum) at $x$ with respect to $\Omega_{k,l} := bar\Omega_k \cup \bar{\Omega}_l$ and if $\varphi$ is $k, l$-differentiable at $x$.*

Additionally to the test function technique, there exists another way of characterizing viscosity solutions. In fact, as the characterization by test functions is of local nature, only their behavior around the point $x$ is of interest. To be more precise, we only require the knowledge of their first derivatives along with the fact that their difference with the tested function $u$ attains an extremum at $x$. It therefore suggests itself to dispense with the test function concept and to replace it by a formulation employing a generalized concept of differentials of continuous functions, the so-called *semijets*.

The motivational idea of semijets is to comprise all possible first derivatives of upper and lower test functions of a function at a given point in two *sets*, called the *upper and lower semijet*, respectively. At a ramification point $x \in \Sigma$ and two incident branches $k, l \in \mathrm{Inc_x}$, the *upper and lower $k, l$-semijets* are defined analogously, where the two branches are thought of as one domain without ramification space.

**Definition 7.22** *Let $u \in C(\Omega)$.*

*(i) Let $x \in \Omega_j$, $j \in J$.*

*We define the* super- (sub-) semijet $J_j^+ u(x)$ $(J_j^- u(x))$ of $u$ at $x$ *to be the set of all $p \in T_x^j \Omega$ such that*

$$u(x) \geq \,(\leq)\, u(y_n) + \langle D^j \boldsymbol{i}_x(x)(p), \boldsymbol{i}_x(y_n) - \boldsymbol{i}_x(x) \rangle + o(d(x, y_n))$$

*for every sequence $y_n \in \Omega_j \cap V_x$ with $y_n \to x$.*

*(ii) Let $x \in \Sigma$ and let $k, l \in Inc_x$.*

*We define the* super- (sub-) $k, l$-semijet $J_{k,l}^+ u(x)$ $(J_{k,l}^- u(x))$ of $u$ at $x$ *to be the set of all $p \in T_x^k \Omega$ such that*

$$u(x) \geq \,(\leq)\, u(y_n) + \langle D^k \boldsymbol{i}_x(x)(p), \boldsymbol{i}_x(y_n) - \boldsymbol{i}_x(x) \rangle + o(d(x, y_n))$$

*for every sequence $y_n \in \Omega_k \cap V_x$ with $y_n \to x$ and such that*

$$u(x) \geq \,(\leq)\, u(y_n) + \langle D^l \boldsymbol{i}_x(x)(S_k^l(p)), \boldsymbol{i}_x(y_n) - \boldsymbol{i}_x(x) \rangle + o(d(x, y_n))$$

*for every sequence $y_n \in \Omega_l \cap V_x$ with $y_n \to x$, where*

$$S_k^l : T_x^k \Omega \to T_x^l \Omega$$

*is defined as $S_k^l(p) := -p_1 \partial_{\nu_l} + \sum_{m=2}^n p_m \partial_m$ with $p := p_1 \partial_{\nu_k} + \sum_{m=2}^n p_m \partial_m$.*

The following proposition establishes the connection between test functions and semijets.

**Proposition 7.3** *Let $\varphi \in C(\Omega)$.*

*(i) We have $J^+_{k,j}\varphi(x) = \{S^k_j(p),\ p \in J^+_{j,k}\varphi(x)\}$ and $J^-_{k,j}\varphi(x) = \{S^k_j(p),\ p \in J^-_{j,k}\varphi(x)\}$ for any $x \in \Sigma$, $j,k \in \mathrm{Inc_x}$, $j \neq k$.*

*(ii) If $\varphi$ is differentiable at $x \in \Omega_j$, $j \in I$, then we have $J^+_j\varphi(x) = J^-_j\varphi(x) = \{D^j\varphi(x)\}$.*

*(iii) If $\varphi$ is $j,k$-differentiable at $x \in \Sigma$, $j,k \in \mathrm{Inc_x}$, $j \neq k$, then we have $J^+_{j,k}\varphi(x) = J^-_{j,k}\varphi(x) = \{D^j\varphi(x)\}$.*

**Proof.** The proof is an easy consequence of the Taylor expansion theorem and the definition of $j,k$-differentiability. □

**Proposition 7.4** *Let $u \in C(\Omega)$.*

*(i) Assume $\varphi \in C(\Omega)$ is an upper (lower) test function of $u$ at $x \in \Omega_j$ for some $j \in J$. Then $D^j\varphi \in J^+_j u(x)$ ($D^j\varphi \in J^-_j u(x)$). On the other hand, assume $p \in J^+_j u(x)$ ($p \in J^-_j u(x)$). Then there is an upper (lower) test function $\varphi \in C(\Omega)$ of $u$ at $x$, which is $C^1$-differentiable in an open neighborhood $U \subset \Omega_j$ of $x$, and which satisfies $p = D^j\varphi(x)$.*

*(ii) Assume $\varphi \in C(\Omega)$ is an upper (lower) $k,l$-test function of $u$ at $x \in \Sigma$ for some $k,l \in \mathrm{Inc_x}$, $k \neq l$. Then $D^k\varphi \in J^+_{k,l}u(x)$ ($D^k\varphi \in J^-_{k,l}u(x)$). On the other hand, assume $p \in J^+_{k,l}u(x)$ ($p \in J^-_{k,l}u(x)$). Then there are an upper (lower) $k,l$-test function $\varphi \in C(\Omega)$ of $u$ at $x$ and an open neighborhood $U \subset \bar{\Omega}_k \cup \bar{\Omega}_l$, such that $\varphi$ is $C^1$-differentiable on $U$ in the sense of definition 7.8 and $k,l$-differentiable on $\Sigma \cap U$, and which satisfies $p = D^k\varphi(x)$.*

**Proof.** This is a direct consequence of the definition of semijets and test functions. □

By means of a semijet characterization we now define viscosity solutions for Hamilton-Jacobi equations on LEP spaces. Note the asymmetry of the definitions of viscosity sub- and supersolutions.

**Definition 7.23** *We say that a function $u \in C(\Omega)$ satisfies the* subsolution condition (for $H$) at $x \in \Omega$ *(or, alternatively, that it formally satisfies*

$$H(Du(x), u(x), x) \leq 0$$

*in the viscosity sense), if the following is the case.*

*(i) If $x \in \Omega_j$ for some $j \in J$, then*

$$H^j(p, u(x), x) \leq 0 \quad \text{for all } p \in J^+_j u(x).$$

*(ii) If $x \in \Sigma$, then for all $k, l \in Inc_x$, $k \neq l$, we have*

$$H^k(p, u(x), x) = H^l(S_k^l(p), u(x), x) \leq 0 \quad \textit{for all } p \in J^+_{k,l}u(x).$$

*We say that a function $u \in C(\Omega)$ satisfies the* supersolution condition (for $H$) at $x \in \Omega$ *(or, alternatively, that it formally satisfies*

$$H(Du(x), u(x), x) \geq 0$$

*in the viscosity sense), if the following is the case.*

*(i) If $x \in \Omega_j$ for some $j \in J$, then*

$$H^j(p, u(x), x) \geq 0 \quad \textit{for all } p \in J^-_j u(x).$$

*(ii) If $x \in \Sigma$, then for all $p^\top \in T_x\Sigma$ and for each $k \in \mathrm{Inc_x}$ there is an index $l \in \mathrm{Inc_x}$, $l \neq k$, such that*

$$H^k(p, u(x), x) = H^l(S_k^l(p), u(x), x) \geq 0 \quad \textit{for all } p \in J^-_{k,l}u(x) \textit{ with } \pi_k^\top(p) = p^\top.$$

*Here $\pi_k^\top : T_x^k\Omega \to T_x\Sigma$ is the projection given by $\pi_k^\top(p) := \sum_{m=2}^n p_m\partial_m$ for $p = p_1\partial_{\nu_k} + \sum_{m=2}^n p_m\partial_m$.*

**Proposition 7.5** *Let $x \in \Omega$ and let $u, v \in C(\Omega)$ satisfy*

$$H(Du(x), u(x), x) \leq 0 \quad \textit{and} \quad H(Dv(x), v(x), x) \leq 0$$

*in the viscosity sense. Then we have $H(Dw(x), w(x), x) \leq 0$ in the viscosity sense for $w(y) := \max\{u(y), v(y)\}$, $y \in \Omega$.*

**Proof.** We restrict ourselves to the case $x \in \Sigma$. Let $k, l \in \mathrm{Inc_x}$, $k \neq l$, and assume there is a $p \in J^+_{k,l}w(x)$, as otherwise there is nothing to prove. Without restriction we also assume that we have $w(x) = u(x)$. By the definition of $w$ and by the definition of $(j, k)$-superjets it follows $p \in J^+_{j,k}u(x)$. As $u$ satisfies the viscosity subsolution condition at $x$, it follows

$$H^k(p, u(x), x) = H^l(S_k^l(p), u(x), x) \leq 0.$$

By $w(x) = u(x)$ it follows

$$H^k(p, w(x), x) = H^l(S_k^l(p), w(x), x) \leq 0.$$

□

### 7.2.2 Existence

As above let $\Omega$ always be an $n$-dimensional LEP space, $n \geq 2$, with ramification space $\Sigma$. In the present section we will show – analogously to chapter 5 – that for a given Hamiltonian $H$ satisfying (7.1) and (7.2) and for any prescribed boundary data $\phi : \partial\Omega \to \mathbb{R}$ we can find a viscosity solution staying below $\phi$ on $\partial\Omega$. The proofs resemble those given in chapter 5 and are based upon the Perron type construction method for viscosity solutions due to Ishii; for the sake of a clear presentation we therefore go through identical parts quickly, whereas different parts will be treated more extensively.

Let $\phi : \partial\Omega \to \mathbb{R}$ be an arbitrary boundary value function and let

$$X := \{u \in C(\Omega) \text{ with } H(Du(x), u(x), x) \leq 0 \text{ in the viscosity sense } \forall x \in \Omega \text{ , } u \leq \phi \text{ on } \partial\Omega\}.$$

**Lemma 7.1** *Let $\emptyset \neq S \subseteq X$ and set $\bar{u}(x) := \sup_{u \in S} u(x)$ for all $x \in \Omega$. Then $\bar{u} \in X$.*

**Proof.** First observe that we clearly have $\bar{u} \leq \phi$ on $\partial\Omega$. It remains to verify that $\bar{u}$ satisfies the viscosity subsolution condition for all $x \in \Omega$. As the case $x \notin \Sigma$ is based upon similar, but easier, arguments, we restrict ourselves to ramification points $x \in \Sigma$.

Let $x \in \Sigma$ and fix $k, l \in \mathrm{Inc_x}$, $k \neq l$. By the definition of $\bar{u}$ there is for each $m \in \mathbb{N}$ a function $u_m \in S$ with $\bar{u}(x) - u_m(x) \leq 1/m$. Now fix $p \in J^+_{k,l}\bar{u}(x)$. We have to show

$$H^k(p, \bar{u}(x), x) \leq 0.$$

To this end let $\varphi$ be an upper $(k,l)$-test function of $\bar{u}$ at $x$ with $D^k\varphi(x) = p$ and $\varphi(x) = \bar{u}(x)$. Furthermore we may assume that $\varphi$ be $C^1$-differentiable in an open neighborhood $V$ of $x$ and that it be $k,l$-differentiable for all $x \in \Sigma \cap V$ (cf. proposition 7.4). Now set

$$d_x(y) := d(x, y) \quad \text{for } y \in V$$

and define the auxiliary function $\varphi_\delta \in C(\Omega)$ by

$$\varphi_\delta(y) := \varphi(y) + \delta(d_x(y))^2$$

for some $\delta > 0$. As $\varphi$ is an upper $(k,l)$-test function of $\bar{u}$ at $x$, the function $\bar{u} - \varphi$ attains a local maximum at $x$. Hence the function $\bar{u} - \varphi_\delta$ also attains a local maximum at $x$. Thus there is a radius $r > 0$ such that the latter maximum is global with respect to the compact set $C := \bar{B}$, where $B := B_r(x) \cap (\bar{\Omega}_k \cup \bar{\Omega}_l)$. For any $m \in \mathbb{N}$ let $x_m \in C$ be a point at which the continuous function

$$y \mapsto u_m(y) - \varphi(y) - 2\delta(d_x(y))^2,$$

attains its maximum with respect to $C$. We show that the sequence $(x_m)_{m\in\mathbb{N}}$ converges to $x$ as $m \to \infty$. Fix $m \in \mathbb{N}$. By the definition of $x_m$ we have

$$u_m(x) - \varphi(x) \leq u_m(x_m) - \varphi(x_m) - 2\delta(d_x(x_m))^2.$$

Using $\bar{u}(x) - u_m(x) < 1/m$ along with $u_m(x_m) \leq \bar{u}(x_m)$ we further conclude

$$\bar{u}(x) - 1/m < \bar{u}(x_m) + \varphi(x) - \varphi(x_m) - 2\delta(d_x(x_m))^2. \tag{7.4}$$

By the fact that $\bar{u} - \varphi_\delta$ attains its global maximum over $C$ at $x$, we obtain

$$\bar{u}(x) - \varphi(x) \geq \bar{u}(x_m) - \varphi(x_m) - \delta(d_x(x_m))^2. \tag{7.5}$$

Subtracting inequality (7.5) from (7.4) yields

$$-1/m + \varphi(x) < \varphi(x) - \delta(d_x(x_m))^2 \quad \text{or} \quad 1/m > \delta(d_x(x_m))^2,$$

implying $x_m \to x$ as $m \to \infty$. This, in turn, means that for all sufficiently large $m \in \mathbb{N}$ the points $x_m$ lie in the *interior* of $C$. We thus truncate $(x_m)$ such that all functions $u_m - \varphi_{2\delta}$ attain local maxima with respect to $\bar{\Omega}_k \cup \bar{\Omega}_l$ at $x_m$, where

$$\varphi_{2\delta}(y) := \varphi(y) + 2\delta(d_x(y))^2.$$

Now we distinguish two cases.

*Case 1.* There are infinitely many $x_m$ with $x_m \notin \Sigma$. Then there is a subsequence – also denoted by $(x_m)$ –, which is completely contained in either $\Omega_k$ or $\Omega_l$. Without restriction we assume $x_m \in \Omega_k$ for all $m \in \mathbb{N}$. As at $x_m$ the function $u_m - \varphi_{2\delta}$ attains a local maximum with respect to $\bar{\Omega}_k \cup \bar{\Omega}_l$ and as the function $\varphi_{2\delta}$ is $C^1$-differentiable, the latter is an upper test function of $u_m$ at $x_m$, implying $D^k\varphi_{2\delta} \in J_k^+ u_m(x_m)$. As $u_m$ satisfies the viscosity subsolution condition at $x_m$, we conclude

$$H^k(D_k\varphi_{2\delta}(x_m), u_m(x_m), x_m) \leq 0. \tag{7.6}$$

Now, since $\varphi$ is $C^1$-differentiable in $V$, we have

$$\lim_{m\to\infty} D^k\varphi_{2\delta}(x_m) = D^k\varphi(x) = p. \tag{7.7}$$

Moreover, the definitions of $x_m$ and $\bar{u}$ imply

$$\bar{u}(x) = \lim_{m\to\infty} u_m(x) \leq \lim_{m\to\infty} u_m(x_m) \leq \lim_{m\to\infty} \bar{u}(x_m) = \bar{u}(x),$$

whence we have $\lim_{m\to\infty} u_m(x_m) = \bar{u}(x)$. This and the relations (7.6), (7.7) imply

$$H^k(p, \bar{u}(x), x) = \lim_{m\to\infty} H^k(D^k\varphi(x_m), \bar{u}(x_m), x_m) \leq 0,$$

as $H$ is continuous.

*Case 2.* By possibly truncating the sequence $(x_m)$ we can assume that $x_m \in \Sigma \cap V$ for all $m \in \mathbb{N}$. As $\Omega$ is an $n$-dimensional LEP space, $V \cap \Sigma$ is a domain in a hyperplane of $\mathbb{R}^n$. Hence we have

$$\partial_{\nu_j} d_x(y) = 0 \quad \text{for all } y \in \Sigma \cap V \text{ and } j \in \mathrm{Inc_x};$$

in particular it follows that $d_x$ is $k,l$-differentiable on $\Sigma \cap V$. As $\varphi$ is also $k,l$-differentiable on $\Sigma \cap V$, the same holds for $\varphi_{2\delta}$. Thus $\varphi_{2\delta}$ is an upper $(k,l)$-test function of $u_m$ at $x_m$ for all $m \in \mathbb{N}$. As $u_m$ satisfies the viscosity subsolution condition at $x_m$, it follows

$$H^k(D^k\varphi_{2\delta}(x_m), u_m(x_m), x_m) \leq 0 \tag{7.8}$$

for all $m \in \mathbb{N}$. We furthermore have

$$\lim_{m\to\infty} D^k\varphi_{2\delta}(x_m) = D^k\varphi(x) = p, \tag{7.9}$$

since $\varphi$ is $C^1$-differentiable on $V$. As above we also have $\lim_{m\to\infty} u_m(x_m) = \bar{u}(x)$. This, relations (7.8) and (7.9), as well as the continuity of $H$ imply

$$H^k(p, \bar{u}(x), x) = \lim_{m\to\infty} H^k(D^k\varphi_{2\delta}(x_m), u_m(x_m), x_m) \leq 0.$$

As the choice of $k$ and $l$ was arbitrary, it follows that $\bar{u}$ satisfies the viscosity subsolution condition at $x$. $\square$

**Lemma 7.2** *Define* $\bar{u}(x) := \sup_{u\in X} u(x)$, $x \in \Omega$. *Then* $\bar{u}$ *satisfies the viscosity supersolution condition for all* $x \in \Omega$.

**Proof.** As in the previous proof we restrict ourselves to verifying the viscosity supersolution condition on $\Sigma$. For this purpose let $x \in \Sigma$ and assume that $\bar{u}$ do *not* satisfy the condition at $x$. Then – according to definition 7.23 –, there is a tangent vector $p^\top \in T_x\Sigma$ and an index $k \in \mathrm{Inc_x}$ such that for all $l \in \mathrm{Inc_x}$, $l \neq k$, there is a tangent vector $p_l \in J^-_{k,l}u(x)$ with $\pi^\top(p_l) = p^\top$ and

$$H^k(p_l, \bar{u}(x), x) < 0. \tag{7.10}$$

Then, according to proposition 7.4, there is an open neighborhood $U$ of $x$ and a family of functions $\varphi_l \in C(\Omega)$, $l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}$, such that for all $l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}$ we have

$$\begin{array}{rl}
(i) & \varphi_l \text{ is } C^1\text{-differentiable on } u \cap (\Omega_k \cup \Omega_l) \\
(ii) & \varphi_l \text{ is } (k,l)\text{-differentiable on } U \cap \Sigma \\
(iii) & D^k\varphi_l(x) = p_l \\
(iv) & \varphi_l(x) = \bar{u}(x) \\
(v) & \varphi_l(y) < \bar{u}(y) \text{ for all } y \in U\backslash\{x\} \\
(vi) & \exists \varphi_\Sigma \in C(U \cap \Sigma) \text{ such that } \varphi_l \equiv \varphi_\Sigma \text{ on } \Sigma.
\end{array} \tag{7.11}$$

Note that we are allowed to assume property (vi), as we have $\pi^\top(D^k\varphi_l) = p^\top$ for all $l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}$.

Due to relation (7.10) and by the continuity of $H$ there is a small number $\delta > 0$ such that for all $l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}$ and all $m \in \mathrm{Inc_x}$ we have

$$H^m(D^m\varphi_l(y), \varphi_l(y), y) < 0$$

for all $y \in B_m := \bar{B}_\delta(x) \cap \bar{\Omega}_m$. We furthermore assume $\delta$ to be sufficiently small to have $B_\delta(x) \subset U$.

Hence, as $B_m$ is compact, there is a number $\varepsilon_0 > 0$ such that for all $l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}$ and all $m \in \mathrm{Inc_x}$ we have

$$H^m(D^m\varphi_l(y), \varphi_l(y) + \varepsilon, y) < 0 \tag{7.12}$$

for all $0 \leq \varepsilon < \varepsilon_0$ and all $y \in B_m$.

We now construct a function $v \in X$ such that $v(x) > \bar{u}(x)$, contradicting the supremal property of $\bar{u}$. In fact, on $\bar{B}_\delta(x)$ we define

$$v(y) := \begin{cases} \max_{l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}} \varphi_l(y) + \varepsilon & \text{if } y \in B_k \\ \varphi_l(y) + \varepsilon & \text{if } y \in B_l\backslash\Sigma \text{ for some } l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}, \end{cases}$$

where $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$ is yet to be determined. Observe that $v$ is countinuous on $\bar{B}_\delta(x)$, as the functions $\varphi_l$ coincide on $\bar{B}_\delta(x) \cap \Sigma$. We show that $v$ satisfies the viscosity subsolution condition at each $y \in B_\delta(x)$, i.e., for each choice of $i, j \in \mathrm{Inc_y}$, $i \neq j$, we have to show

$$H^i(p, v(y), y) \leq 0 \tag{7.13}$$

for all $p \in J^+_{i,j}v(y)$.

*Case 1.* Let $y \in B_\delta(x) \cap \Sigma$. Choose

$$p = p_1\partial_{\nu_i} + \sum_{m=2}^{n} p_m\partial_m \in J^+_{i,j}v(y)$$

arbitrarily. Let us first assume the special case $i \neq k$ and $j \neq k$. By the definition of $v$ and by the definition of superdifferential it follows

$$p_m = \partial_m\varphi_i(y) \text{ for all } m = 2, \ldots, n \text{ and } p_1 \geq \partial_{\nu_i}\varphi_i(y).$$

This implies

$$|p| \leq |D^i\varphi_i(y)|, \text{ provided we have } p_1 \leq 0. \tag{7.14}$$

Applying the same arguments to

$$S^j_i(p) = -p_1 d\nu_j + \sum_{j=2}^{n} p_j dx^j \in J^+_{j,i}v(y),$$

we derive $-p_1 \geq \partial_{\nu_j}\varphi_j(y)$, implying

$$|S_i^j(p)| \leq |D^j\varphi_j(y)|, \text{ provided we have } -p_1 \leq 0. \tag{7.15}$$

Then (7.14) and (7.15) yield

$$|p| = |S_i^j(p)| \leq \max\{|D^i\varphi_i(y)|, |D^j\varphi_j(y)|\}. \tag{7.16}$$

Now observe that conditions (7.1) and (7.2) imply that for $l = i, j$ the functions

$$h^l : T_y^l\Omega \to \mathbb{R},\ q \mapsto H^l(q, v(y), y)$$

are strictly convex and rotationally symmetric in $q$. Consequently, $h^l(q)$, $l = i, j$, is strictly isotonic in $|q|$. By (7.16) we thus have

$$H^i(p, v(y), y) = H^j(S_i^j(p), v(y), y) \leq \max\{H^i(D^i\varphi_i(y), v(y), y), H^j(D^j\varphi_j(y), v(y), y)\}.$$

Hence (7.13) follows upon choosing $l = m = i$ and $l = m = j$, respectively, in (7.12).

To prove the general case it remains to consider the situation where one of the indices $i, j$, say $i$, coincides with $k$. Then by the definition of $v$ and by the definition of superdifferential we have

$$p_1 \leq \max\{\partial_{\nu_i}\varphi_i(y), \partial_{\nu_i}\varphi_j(y)\} \text{ and } p_m = D^m\varphi_i(y) = D^m\varphi_j(y) \text{ for all } m = 2, \ldots, n.$$

This implies

$$|p| \leq \max\{|D^i\varphi_i(y)|, |D^i\varphi_j(y)|\}, \text{ provided } p_1 \geq 0.$$

We combine this with (7.15), which derive as above, in order to obtain (7.16). We then proceed as above to show (7.13).

*Case 2.* Let $y \in B_\delta(x)\backslash\Sigma$. If $y \in \Omega_l$, $l \neq k$, (7.13) follows immediately from (7.12) and the definition of $v$. If $y \in \Omega_k$, (7.12) implies that each $\varphi_l$, $l \in \mathrm{Inc_x}\backslash\{\mathrm{k}\}$, satisfies the viscosity subsolution condition at $y$. Then (7.13) follows by the definition of $v$ and proposition 7.5.

By (7.11) (v) we may now choose $\varepsilon$ with $0 < \varepsilon < \varepsilon_0$ such that $v(y) < \bar{u}(y)$ for all $y \in \partial B_\delta(x)$. It follows that the function $w : \Omega \to \mathbb{R}$ given by

$$w(y) := \begin{cases} \max\{\bar{u}(y), v(y)\} & \text{if } y \in B_\delta(x) \\ \bar{u}(y) & \text{if } y \notin B_\delta(x) \end{cases}$$

is continuous on $\Omega$. Furthermore, by (7.13) and by $\bar{u} \in X$ we invoke proposition 7.5 to conclude that $w$ satisfies the viscosity subsolution condition on $B_\delta(x)$. Hence, as $w \equiv \bar{u}$ on $\Omega\backslash B_\delta(x)$, we have $w \in X$. Finally observe that we have $w(x) > \bar{u}(x)$. Thus we have derived a contradiction to the definition of $\bar{u}$, which completes the proof. □

### 7.2.3 Uniqueness

We start with the following comparison result.

**Lemma 7.3** *Let $f \in C(\bar{\Omega})$ with $f < 0$ on $\bar{\Omega}$. Furthermore let $u, v \in C(\bar{\Omega})$ satisfy*

$$H(Du(x), u(x), x) \leq f(x) \quad \text{and} \quad H(Dv(x), v(x), x) \geq 0$$

*in the viscosity sense for all $x \in \Omega$. Furthermore assume $u \leq v$ on $\partial\Omega$. Then $u \leq v$ on $\Omega$.*

**Proof.** We assume the contrary, i.e., that there is a point $z \in \Omega$ with $u(z) > v(z)$. For $\varepsilon > 0$ we define the continuous function

$$\Phi : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R} \quad \text{by} \quad \Phi(x, y) := u(x) - v(y) + 5M\beta(\varepsilon^{-1} d(x, y))$$

with $M := \max_{x \in \bar{\Omega}} \max\{u(x), v(x)\}$ and $\beta \in C^\infty(\mathbb{R})$ with $0 \leq \beta \leq 1$, $\beta(0) = 1$, $\beta(x) = 0$ for all $|x| \geq 1$ and $\beta'(0) = 0$.

Let $(p, q) \in \bar{\Omega} \times \bar{\Omega}$ be a point where $\Phi$ attains its global maximum with respect to $\bar{\Omega} \times \bar{\Omega}$.

We show that we can achieve

$$(p, q) \in \Omega \times \Omega \tag{7.17}$$

by choosing $\varepsilon > 0$ sufficiently small. To this end set

$$m_\varepsilon := \sup\{u(x) - v(y) \mid d(x, y) \leq \varepsilon, (x, y) \in (\bar{\Omega} \times \bar{\Omega}) \backslash (\Omega \times \Omega)\}$$

and observe that $\limsup_{\varepsilon \to 0} m_\varepsilon \leq 0$ as $\varepsilon \to 0$ due to continuity reasons and by the fact that we have $u \leq v$ on $\partial\Omega$. Now choose $\varepsilon > 0$ so small that we have $m_\varepsilon < u(z) - v(z)$. Then $\Phi(z, z) > \Phi(x, y)$ for any choice $(x, y) \in (\bar{\Omega} \times \bar{\Omega}) \backslash (\Omega \times \Omega)$ with $d(x, y) \leq \varepsilon$. Consequently we have $(p, q) \in \Omega \times \Omega$.

We furthermore have

$$d(p, q) \leq \varepsilon. \tag{7.18}$$

In fact, if this were not the case, we would have

$$\Phi(z, z) - \Phi(p, q) = u(z) - v(z) + 5M - u(p) - v(q) > 0$$

by the definition of $M$, a contradiction to the choice of $(p, q)$.

Let $m := \max_{\bar{\Omega}} -f > 0$ and

$$M_0 := \max\{|D^l H^l(p, u(x), x)|,\ x \in \bar{\Omega}_l, l \in J\} < +\infty,$$

where the latter exists due to (7.3). Choosing

$$\varepsilon < m/(2M_0) \tag{7.19}$$

sufficiently small then reduces the situation to the following different cases.

*Case 1.* There are $j, k \in J$, $j \neq k$, such that $p \in \Omega_j$ and $q \in \Omega_k$. Then the functions $\varphi_q$ and $\varphi_p$ are continuously differentiable at $p$ and $q$, respectively, where

$$\varphi_q : \Omega \to \mathbb{R}, \quad \varphi_q(x) := 5M\beta(\varepsilon^{-1}d_q(x)) \quad \text{and} \quad \varphi_p : \Omega \to \mathbb{R}, \quad \varphi_p : x \mapsto 5M\beta(\varepsilon^{-1}d_p(x)).$$

Here $d_y := d(\cdot, y)$ for any $y \in \Omega$. Observe furthermore that by the choice of $p$ and $q$ the function $u + \varphi_q$ attains a local maximum at $p$, whereas the function $v - \varphi_p$ attains a local minimum at $q$. Hence $-\varphi_q$ is an upper test function of $u$ at $p$ and $\varphi_p$ is a lower test function of $v$ at $q$. It follows

$$-D^j\varphi_q(p) = -5M\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)) \cdot D^j d_q(p) \in J_j^+ u(p) \tag{7.20}$$

and

$$D^k\varphi_p(q) = 5M\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q)) \cdot D^k d_p(q) \in J_k^- v(q). \tag{7.21}$$

By the properties of $u$ and $v$ we respectively obtain

$$f(p) > H^j(-D^j\varphi_q(p), u(p), p) \tag{7.22}$$

and

$$H^k(D^k\varphi_p(q), v(q), q) > 0. \tag{7.23}$$

By 7.18 there is a point $x \in \Sigma$ such that $d(x,p) \leq \varepsilon$ and $d(x,q) \leq \varepsilon$. Without restriction we can assume that there is a canonical identification chart $(V_x, \mathbf{i}_x)$ around $x$ with $p, q \in V_x$; furthermore assume that the straight lines connecting $x$ and $p$ as well as $x$ and $q$, respectively, are contained in $V_x$. We set

$$\tilde{p} := \mathbf{i}_x(p),\ \tilde{q} := \mathbf{i}_x(q),\ \tilde{x} := \mathbf{i}_x(x),\ \tilde{u} := u \circ \mathbf{i}_x^{-1},\ \tilde{v} := v \circ \mathbf{i}_x^{-1} \tag{7.24}$$

and

$$\eta_p := D^j\mathbf{i}_x(p)(D^j\varphi_q(p)),\ \eta_q := D^k\mathbf{i}_x(q)(D^k\varphi_p(q)).$$

By the mean value theorem there are points

$$\xi_p \in \mathcal{R}^n_{r(x),j} \text{ and } \xi_q \in \mathcal{R}^n_{r(x),k}$$

lying on the respective straight lines connecting $\tilde{p}$ and $\tilde{x}$ and $\tilde{q}$ and $\tilde{x}$ such that

$$\begin{aligned} H^j(-D^j\varphi_q(p), u(p), p) &= \tilde{H}^j(-\eta_p, \tilde{u}(\tilde{p}), \tilde{p}) = D_x\tilde{H}^j(-\eta_p, \tilde{u}(\tilde{p}), \xi_p)(\tilde{p} - x) + \tilde{H}^j(-\eta_p, \tilde{u}(\tilde{p}), \tilde{x}) \\ &> \tilde{H}^j(-\eta_p, \tilde{u}(\tilde{p}), \tilde{x}) - M_0\varepsilon \end{aligned} \tag{7.25}$$

and

$$\begin{aligned} H^k(D^k\varphi_p(q), v(q), q) &= \tilde{H}^k(\eta_q, \tilde{v}(\tilde{q}), \tilde{q}) = D_x\tilde{H}^k(\eta_q, \tilde{v}(\tilde{q}), \xi_q)(\tilde{q} - \tilde{x}) + \tilde{H}^k(\eta_q, \tilde{v}(\tilde{q}), \tilde{x}) \\ &< \tilde{H}^k(\eta_q, \tilde{v}(\tilde{q}), \tilde{x}) + M_0\varepsilon. \end{aligned} \tag{7.26}$$

By the properties of the distance function we conclude

$$|D^j\mathbf{i}_x(p)(D^jd_q(p))| = |D^k\mathbf{i}_x(q)(D^kd_p(q))| = 1,$$

whence we get

$$|\eta_q| = 5M\varepsilon^{-1}|\beta'(\varepsilon^{-1}d(p,q))| = |\eta_p|.$$

By (7.2) we therefore have

$$\tilde{H}^j(-\eta_p, \tilde{u}(\tilde{p}), x) = H^k(\eta_q, \tilde{u}(\tilde{p}), x). \tag{7.27}$$

The definition of $\Phi$ yields

$$u(p) - v(q) + 5M\beta(\varepsilon^{-1}d(p,q)) = \Phi(p,q) \geq \Phi(z,z) = u(z) - v(z) + 5M > 5M,$$

whence we have $\tilde{u}(\tilde{p}) = u(p) > v(q) = \tilde{v}(\tilde{q})$. By virtue of (7.1) we then obtain

$$\tilde{H}^k(\eta_q, \tilde{u}(\tilde{p}), \tilde{x}) \geq \tilde{H}^k(\eta_q, \tilde{v}(\tilde{q}), \tilde{x}). \tag{7.28}$$

If we successively arrange the relations (7.22), (7.25), (7.27), (7.28), (7.26), and (7.23), we finally obtain

$$-m > f(p) > -2M_0\varepsilon,$$

a contradiction to (7.19).

*Case 2.* There is a $j \in J$ such that $p, q \in \Omega_j$. Then there is a point $x \in \Omega_j$ such that for the canonical identification chart $(V_x, \mathbf{i}_x)$ around $x$ we have $p, q \in V_x$. Let us assume (7.24) and set

$$\eta_p := D^j\mathbf{i}_x(p)(D^j\varphi_q(p)),\ \eta_q := D^j\mathbf{i}_x(q)(D^j\varphi_p(q)).$$

Similarly to case 1 we derive

$$f(p) > H^j(-D^j\varphi_q(p), u(p), p) = \tilde{H}^j(-\eta_p, \tilde{u}(\tilde{p}), \tilde{p}) \tag{7.29}$$

and

$$0 < H^j(D^j\varphi_p(q), v(q), q) = \tilde{H}^j(\eta_q, \tilde{v}(\tilde{q}), \tilde{q}). \tag{7.30}$$

By the properties of the distance function we have

$$D^j\mathbf{i}_x(p)(D^jd_q(p)) = -D^j\mathbf{i}_x(q)(D^jd_p(q)),$$

which implies

$$\begin{aligned}\eta_p &= 5M\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q))D^j\mathbf{i}_x(p)(D^jd_q(p)) \\ &= -5M\varepsilon^{-1}\beta'(\varepsilon^{-1}d(p,q))D^j\mathbf{i}_x(p)(D^jd_p(q)) = -\eta_q.\end{aligned}$$

Hence

$$\tilde{H}^j(-\eta_p, \tilde{u}(\tilde{p}), \tilde{p}) = \tilde{H}^j(\eta_q, \tilde{u}(\tilde{p}), \tilde{p}). \tag{7.31}$$

We also have $u(p) > v(q)$, whence

$$\tilde{H}^j(\eta_q, \tilde{u}(\tilde{p}), \tilde{p}) \geq \tilde{H}^j(\eta_q, \tilde{v}(\tilde{q}), \tilde{p}). \tag{7.32}$$

The mean value theorem finally yields

$$\begin{aligned} \tilde{H}^j(\eta_q, \tilde{v}(\tilde{q}), \tilde{p}) &= D_x\tilde{H}^j(\eta_q, \tilde{v}(\tilde{q}), \xi)(\tilde{p} - \tilde{q}) + \tilde{H}^j(\eta_q, \tilde{v}(\tilde{q}), \tilde{q}) \\ &\geq \tilde{H}^j(\eta_q, \tilde{v}(\tilde{q}), \tilde{q}) - M_0\varepsilon \end{aligned} \tag{7.33}$$

for some $\xi \in \mathbb{R}^n$ on the straight line connecting $\tilde{p}$ and $\tilde{q}$. The successive combination of (7.29), (7.31), (7.32), (7.33), and (7.30), yields

$$-m > f(p) > -M_0\varepsilon, \tag{7.34}$$

a contradiction to (7.19).

*Case 3.* We have $q \in \Sigma$ and there is a $j \in \mathrm{Inc_p}$ such that $p \in \Omega_j$. We may assume that there is a canonical identification chart $(V_q, \mathbf{i}_q)$ with $p \in V_q$. We first derive (7.22) similarly to case 1. Now observe that for the function $d_p$ we have

$$\partial_{\nu_j} d_p(q) = -\partial_{\nu_k} d_p(q) \quad \text{for all } k \in \mathrm{Inc_x}\backslash\{\mathrm{j}\},$$

which implies that $d_p$ is $j,k$-differentiable at $p$ for all $k \in \mathrm{Inc_x}\backslash\{\mathrm{j}\}$. It immediately follows that the same holds for $\varphi_p$. Therefore and in view of the fact that $v - \varphi_p$ attains a local minimum at $q$, we have

$$D^j\varphi_p(q) \in J^-_{j,k}v(q) \quad \text{for all } k \in \mathrm{Inc_x}\backslash\{\mathrm{j}\}. \tag{7.35}$$

As $v$ satisfies the viscosity supersolution property at $q$, according to definition 7.23 there is an index $k \in \mathrm{Inc_x}\backslash\{\mathrm{j}\}$ such that

$$H^j(D^j\varphi_p(q), v(q), q) \geq 0. \tag{7.36}$$

Now we proceed exactly as in case 2 in order to derive the contradiction (7.34) from (7.22) and (7.36).

*Case 4.* We have $q \in \Sigma$ and there is a $j \in \mathrm{Inc_p}$ such that $p \in \Omega_j$. This case can be treated almost exactly as case 3; note that the only difference is caused by the asymmetry of the definition of viscosity sub- and supersolutions in definition 7.23. However, the definition of subsolutions, which is to be applied in the present case, is less strict than the definition of supersolutions, and it can be readily seen that we once more can derive the contradiction (7.19).

*Case 5.* $p, q \in \Sigma$. Then we can without restriction assume that there is a canonical identification chart $(V_p, \mathbf{i}_p)$ around $p$ with $q \in V_p$. Note that we then have

$$\partial_{\nu_j} d_p(q) = \partial_{\nu_j} d_q(p) = 0 \quad \text{for all } k \in \mathrm{Inc_x}.$$

Hence $\varphi_q$ and $\varphi_p$ are $j,k$-differentiable at $p$ and $q$, respectively, for all $j,k \in \mathrm{Inc_p}$, $j \neq k$. Thus for any fixed $j \in \mathrm{Inc_p}$ we have

$$D^j \varphi_q(p) \in J^+_{j,k} u(p) \quad \text{and} \quad D^j \varphi_p(q) \in J^-_{j,k} v(q)$$

for all $k \in \mathrm{Inc_p} \backslash \{\mathrm{j}\}$. Fix $j \in \mathrm{Inc_p}$. As $u$ satisfies the viscosity subsolution property at $p$, by definition 7.23 we find that we have

$$f(p) \geq H^j(D^j \varphi_q(p), u(p), p).$$

Furthermore, as $v$ satisfies the viscosity supersolution property at $q$, by definition 7.23 we find that we have

$$0 \leq H^j(D^j \varphi_p(q), v(q), q).$$

As the straight line connecting $p$ and $q$ is contained in $\bar{\Omega}$

$\square$

# Bibliography

[1] K.P. Hadeler, C. Kuttler, and I. Gergert. Dirichlet and obstacle problems for granular matter. *SFB 382, Univ. Tübingen, Report*, 173, 2002.

[2] P. Aviles and Y. Giga. The distance function and defect energy. *Proceedings of the Royal Society of Edinburgh*, 126A:923–938, 1996.

[3] M. G. Crandall and P.-L. Lions. Condition d'unicité pour les solutions generalisées des équations de hamilton-jacobi de premier order. *C. R. Acad. Sci. Paris*, 292:183–186, 1981.

[4] M. G. Crandall and P.-L. Lions. Viscosity solutions of hamilton-jacobi equations. *Trans. Amer. Math. Soc.*, 277:1–42, 1983.

[5] S. N. Kružkov. Generalized solutions of the hamilton-jacobi equations of eikonal type. i. formulation of the problems; existence, uniqueness and stability theorems; some properties of the solutions. *Math. USSR Sbornik*, 27:406–446, 1975.

[6] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, 27:1–67, 1992.

[7] S. N. Kružkov. First order quasilinear equations in several idependent variables. *Math. USSR Sbornik*, 10:217–243, 1970.

[8] R. Jensen. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rat. Mech. Anal.*, 101:1–27, 1988.

[9] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order.* Springer Verlag, 1977.

[10] G. F. Roach. *Green's functions - Introductory theory with applications.* Van Nostrand Reinhold Company, 1970.

[11] H. Ishii. Perron method for viscosity solutions. *noch nicht klar*, ?, ?

[12] G. Lumer. Espaces ramifiés et diffusions sur les réseaux topologiques. *C. R. Acad. Sc. Paris, Série A*, 291:627–630, 1980.

[13] S. Nicaise. Polygonal interface problems. *Series "Methoden und Verfahren der Mathematischen Physik"*, 39, 1993.

[14] M. G. Slinko and K. Hartmann. *Methoden und Programme zur Berechnung chemischer Reaktoren*. Akademie-Verlag, Berlin, 1972.

[15] H. Camerer. *Die elektrotonische Ausbreitung im Soma, Dendritenbaum und Axon von Nervenzellen*. Dissertation, Tübingen, 1980.

[16] J. von Below and S. Nicaise. Dynamical interface transition in ramified media with diffusion. *Comm. in Part. Diff. Eq.*, 21:255–279, 1996.

[17] S. Nicaise. Elliptic operators on elementary ramified spaces. *Integral Equations and Operator Theory*, 11:230–257, 1988.

[18] F. Harary. *Graph theory*. Addison-Wesley Publishing Company, 1969.

[19] H. Ishii. A simple, direct proof of uniqueness for solutions of the hamilton-jacobi equations of eikonal type. *Proc. Amer. Math. Soc.*, 100,2, 1987.

[20] K.P. Hadeler and C. Kuttler. Dynamical models for granular matter. *Granular Matter*, 2:9–18, 1999.

[21] K.P. Hadeler and C. Kuttler. Granular matter in a silo. *Granular Matter*, 3:193–197, 2001.

[22] T.H. Cormen, C.E. Leiserson, and R.L. Rivest. *Introduction to algorithms*. MIT Press, Cambridge, 1990.

[23] E.W. Dijkstra. A note on two problems in connexion with graphs. *Numerische Mathematik*, 1:269–271, 1959.

[24] R.K. Ahuja, K. Mehlhorn, J.B. Orlin, and R.E. Tarjan. Faster algorithms for the shortest way problem. *Journal of the ACM*, 37:213–223, 1990.

[25] R. Bellman. On a routing problem. *Quarterly of applied mathematics*, 16(1):87–90, 1958.

[26] L.R. Ford and D.R. Fulkerson. *Flows in networks*. Princeton University Press, 1962.

[27] M. Thorup. Undirected single-source shortest paths with positive integer weights in linear time. *Journal of the ACM*, 46(3):362–394, 1999.