Convexity of Surfaces Moving by Mean Curvature Flow

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Contents

1 Introduction

In this work we consider hypersurfaces moving by their mean curvature in general Riemannian manifolds. In particular, we focus on convex surfaces and examine their behaviour in view of the question under which circumstances convexity is preserved.

Originally, mean curvature flow has been studied by Brakke [1] from the viewpoint of geometric measure theory. Other authors have considered evolutionary surfaces of prescribed mean curvature, such as Ecker [3] or Gerhardt [5]. This research was partly motivated by practical reasons: A typical physical process which can be modelled by evolutionary surfaces of prescribed mean curvature is the behaviour of grain boundaries when pure metal is annealed.

Mean curvature flow of is a special case among a variety of more general deformation laws for hypersurfaces. The general case is given by a one-parameter family $F : M \times [0, T] \to (N, \bar{g})$ of hypersurfaces $M_t = F(\cdot, t)(M)$ satisfying the initial value problem

$$\begin{aligned} \frac{\partial F}{\partial t}(p,t) &= f\nu(p,t), \quad p \in M,\ t \in [0,T], \\ F(\cdot,0)(M) &= M_0. \end{aligned} \tag{1}$$

Here $\nu(p, t)$ is a choice of unit normal at $F(p, t)$ and $F(\cdot, 0)$ a smooth isometric immersion of M into N. Moreover, f is a homogeneous, symmetric functions $f = f(\kappa_1, \dots, \kappa_n)$ of the principal curvatures $\kappa_1, \dots, \kappa_n$ of M^n at p governing the velocity by which M_t is moved along its unit normal at p. In the special case of mean curvature flow one chooses f to be the negative mean curvature, that is $f = -H = -\sum_{i=1}^n \kappa_i$.

Other interesting choices of f yield the inverse mean curvature flow ($f = (\sum_{i=1}^n \kappa_i)^{-1}$), the harmonic mean curvature ($f = -(\sum_{i=1}^n \kappa_i^{-1})^{-1}$) and the Gauss curvature flow ($f = -(\kappa_1 \cdots \kappa_n)$). Studying these flows is important in view of the various applications both in differential geometry and mathematical physics such as general relativity. Techniques both from differential geometry and the theory of partial differential equations are required to deal with the topic.

Among the above flows, mean curvature flow has been most completely understood. This is largely due to the fact that it can be expressed by a quasilinear parabolic system of second order. One therefore has the possibility of applying parabolic techniques such as the maximum principle, which is a powerful means to give global statements about the flow.

In 1984, Huisken [7] has thoroughly discussed the mean curvature flow of closed (i.e. compact without boundary) initial hypersurfaces M_0, which are smoothly immersed in Euclidean space and are at least two-dimensional. Such a surface M_0 is called *uniformly convex*, if all eigenvalues of its second fundamental form are strictly positive everywhere. We then write

$$h_{ij} > 0 \quad \text{everywhere on } M_0. \tag{2}$$

Basically, Huisken has shown that uniformly convex surfaces contract to single points in a finite time interval while more and more taking the shape of a sphere. As an important minor result of his work one obtains that convexity of the surface is preserved during the flow - a topic we will particularly elaborate on in this paper.

Huisken extended his considerations to general Riemannian ambient spaces in his 1986 paper [8]. These results reveal a very direct interplay between the geometric properties of the underlying ambient space N and the behaviour of the evolving hypersurface M_t. In contrast to Euclidean ambient space, the preservation of convexity can not be taken for granted anymore. Rather the notion of convexity has to be adapted to this setting, taking into account bounds of the ambient curvature. To be precise, the initial surface M_0 now has to satisfy the condition

$$Hh_{ij} > nK_1 g_{ij} + \frac{n^2}{H} L g_{ij} \quad \text{everywhere on } M_0, \tag{3}$$

where $-K_1 \leq 0$ is a uniform lower bound for the sectional curvatures of N, and $L \geq 0$ is a bound for the gradient of the ambient Riemannian curvature tensor with $|\bar{\nabla}\bar{R}|^2 \leq L^2$. Only a preservation of this "modified" convexity can be guaranteed in the general Riemannian setting.

Our goal in the present paper is to discuss Huisken's theorem concerning the preservation of convexity as given in [8]. For this purpose we not only present its proof along with a detailed introduction to the topic, but also discuss its rigidity by giving two examples.

The paper is therefore basically divided into two parts: In the first part we develop the preliminary theory and afterwards prove the convexity theorem. Additionally we devote an extra chapter to another result of [8]: The pinching of the principal curvatures. This is an important statement on its own, which provides another very interesting piece of information about the behaviour of the second fundamental form and thus has been chosen to be included. As we are interested in the influence of the ambient curvature, we base all considerations on the general Riemannian setting. In [8], however, this general case occurs as a generalization of the Euclidean setting treated in [7], which is why much of our work consists of discussing the details which Huisken has omitted in [8].

In the second part of the paper we deal with the question if the convexity theorem proved in the first part might be improved. We give two examples suggesting that it is inevitable to modify the notion of convexity in order to guarantee its preservation. Each of them focusses on the respective occurrence of the additional term involving K_1 and L in (3). In each case we choose ambient spaces with either $K_1 > 0$ and $L = 0$ (first case) or $K_1 = 0$ and $L > 0$ (second case). Then we show that there can be constructed a smooth initial hypersurface M_0 which is convex in the sense of (2), but which loses its convexity immediately when moved by mean curvature flow.

In the first example we consider rotational 2-surfaces in three-dimensional hyperbolic space, which has negative sectional curvature ($K_1 > 0$) but is locally symmetric ($L = 0$). We construct a compact convex hypersurface without boundary losing convexity as soon as moved by mean curvature flow.

In the second example we give a *local* description of a hypersurface which is convex at least on a small domain but loses convexity when moved by mean curvature flow. For this purpose we choose a family of distorted hyperspheres, the *Berger-spheres*, as ambient spaces and use a graph approach to locally construct 2-submanifolds. We demonstrate that in a neighbourhood around some point p we get the desired behaviour by suitably fixing the first few members of the Taylor series of the graph function.

As a preparation for this we describe the Hopf map and the Lie group structure of the 3-sphere, both of which motivate the definition of Berger spheres. Moreover, we provide several important facts from the theory of Lie groups.

The two examples strongly suggest that Huisken's convexity theorem cannot be improved and that both the bounds of the ambient curvature and its gradient have to be taken into account, such that the preservation of (3) is basically the best one can expect.

1.1 Overview

In chapter 2 we briefly introduce the basic concepts of differential geometry focussed on isometric immersions and the second fundamental form. Moreover we derive Simons' identity for the Laplacian of h_{ij}, a tool we later require for the derivation of evolution equations.

In chapter 3 we give an introduction to the concept of mean curvature flow of hypersurfaces in general Riemannian manifolds. We briefly discuss short-time existence and derive several evolution equations of important quantities.

In chapter 4 we discuss the crucial theorem of [8] stating that the modified notion of convexity (3) is preserved under mean curvature flow. We arrange the proof in such a way that the reasons for this modification of convexity get revealed.

In chapter 5 we consider another aspect of the second fundamental form: We present Huisken's proof showing that the principal curvatures of the moving surface approach each other - they get 'pinched' in the end. In a certain sense this is an extra chapter; the following chapters only refer to chapters 3 and 4.

In chapter 6 we discuss the question about possible improvements of the convexity theorem. We use three-dimensional hyperbolic space as ambient space and construct a convex rotational surface losing convexity in the sense of (2).

In chapter 7 we introduce the Berger spheres and its properties. We describe the Hopf map and develop basic parts of Lie group theory. Using Berger spheres as ambient space we finally give a local example of a convex hypersurface losing convexity.

In chapter 8 we give a few concluding remarks and interpretations.

1.2 Acknowledgements and remarks

I would like to mention that throughout this paper I have put great emphasis on motivating the steps and methods used in the proofs. I did not shorten or compress Huisken's proofs. Rather I sometimes changed the order of the arguments, in cases where in my opinion underlying ideas get revealed in a clearer way. Consequently I do not collect auxiliary lemmas before proving a theorem, but in most cases I prove them at the very spots they are required. However, if in certain cases the proof of an auxiliary fact would interrupt the flow of arguments too much, I refer to the appendix, where the proof is given in detail. In cases where a result just proved will be needed only later in the text, I always give a reference about when and where it will be applied.

Finally I remark that I have decided to use the English language to write this thesis, because the topic was suggested to me by Dr Ben Andrews at the Australian National University in Canberra, Australia. At this spot I would like to thank him as well as everyone else at the Department of Mathematics of the ANU for their hospitality and the warm and friendly atmosphere at the institute. Moreover I want to thank Prof Huisken for making my stay in Australia possible as well as his patience. Finally I would like to thank my parents and Yvonne.

2 Basic differential geometrical notions and facts

This chapter provides fundamental definitions and rapidly introduces the reader to the basic facts and notions of differential geometry. Although there is nothing really new for those familiar with the material, one should carefully note which topics we pick out for later purposes.

2.1 Riemannian manifolds

The basic mathematical objects we will work with are Riemannian manifolds of finite dimension, that is sets which locally look like Euclidean space. Let (M, g) be such a Riemannian manifold. The metric g determines the unique *Levi-Civita connection* ∇ on M, which is given by the Christoffel symbols. In terms of a coordinate frame $\{e_i\}$ they are given by

$$\Gamma_{ij}^k := \frac{1}{2} g^{km} \left(e_i(g_{jm}) + e_j(g_{im}) - e_m(g_{ij}) \right).$$

The connection makes possible to *covariantly differentiate* vector fields and tensor fields. A vector field $v = v^i e_i$ given in terms of a coordinate frame $\{e_i\}$ is covariantly differentiated according to

$$\nabla_i v^j = e_i(v^j) + \Gamma_{ik}^j v^k.$$

Note that we assume the Einstein summation rule to hold, that is, we sum over repeated indices.

By preserving the product rule the concept of covariant differentiation is naturally extended to arbitrary mixed tensors of the form $T_{i_1,\ldots,i_n}^{j_1,\ldots,j_m}$ by

$$\nabla_k \left(T_{i_1,\ldots,i_n}^{j_1,\ldots,j_m} \right) = e_k(T_{i_1,\ldots,i_n}^{j_1,\ldots,j_m}) - \sum_{l=1}^{n} \Gamma_{j i_l}^s T_{i_1,\ldots,i_{l-1},s,i_{l+1},\ldots,i_n}^{j_1,\ldots,j_m} + \sum_{l=1}^{m} \Gamma_{js}^{j_l} T_{i_1,\ldots,i_n}^{j_1,\ldots,j_{l-1},s,j_{l+1},\ldots,j_m}.$$

The notion of covariant derivative, in turn, gives rise to the concept of the *Riemannian curvature tensor* R, which for vector fields X, Y, Z is defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Here R acts as a tensor of type $(3, 1)$. In a given coordinate frame $\{e_i\}$ we set

$$R(e_i, e_j)e_k := R_{ij}{}^m{}_k e_m \quad \text{and} \quad g(R(e_i, e_j)e_k, e_m) := R_{ijmk}, \tag{4}$$

where the latter relation manifests R as a $(4, 0)$-tensor.

Furthermore, computing the definition of $R(e_i, e_j)e_k$ by means of the Christoffel symbols Γ_{ij}^k, one gets

$$R_{ij}{}^m{}_k = e_i(\Gamma_{jk}^m) - e_j(\Gamma_{ik}^m) + \Gamma_{jk}^l \Gamma_{il}^m - \Gamma_{ik}^l \Gamma_{jl}^m. \tag{5}$$

2.2 Isometric immersions

In this section we describe isometric immersions as a way how a Riemannian manifold M inherits the geometry of another *ambient* Riemannian manifold N of higher dimension.

Let $(N, \bar{g})$ be an $n+1$-dimensional Riemannian manifold. Now let M be an n-dimensional manifold (smoothly) immersed in N. In other words, we have a diffeomorphism F :

$M \to N$ such that the linear mapping $DF(p) : T_pM \to T_{F(p)}N$ given by the differential $DF : TM \to TN$ is injective for all $p \in M$. M is then called a *hypersurface*.

We want to endow M with some Riemannian metric such that M behaves geometrically the same way as $F(M)$ does. To given vector fields $X, Y \in TF(M) \subset TN$ we therefore assign a metric g by

$$g(q)(X(q), Y(q)) := \bar{g}(q)(X(q), Y(q)).$$

Thus the metric of M is determined by the geometry of N and the way $F(M)$ is situated in N, and M along with this *induced* metric g makes F a (local) isometry. We then say that $M \subset N$ is a (smooth) isometric hypersurface immersion.

For reasons of simplicity we may identify M and $F(M)$ as well as TM and $TF(M)$ as far as geometrical topics are regarded. In the following we will use this identification without explicit mentioning. Moreover we use the convention that all geometric quantities referring to N be indicated by a bar whereas those belonging to M look as usual (in analogy to $\bar{g}$ and g). In particular we distinguish between the two connections ∇ and $\bar{\nabla}$ as well as the curvature tensors R and $\bar{R}$.

Apart from the *intrinsic* geometric quantities of M derived from the induced metric such as ∇ or R, the way how M is situated in N gives rise to another kind of geometric information - the *extrinsic* geometry, which is manifested by the second fundamental form.

2.3 The second fundamental form

Let $F : M \to N$ be a smooth isometric hypersurface immersion. Furthermore assume M to be oriented. Around $p \in M$ we consider a so-called *adapted frame* $e_0, \dots, e_n$ which consists of a local frame $e_1, \dots, e_n$ around p tangent to M and a choice of a unit normal field $e_0 = \nu$ of M (there are, of course, two possibilities, depending on the orientation of M). As the derivative of the Gauss map, which takes M to the n-sphere, we consider the *Weingarten map* $A : T_pM \to T_pM$ and its corresponding bilinear mapping $h : T_pM \times T_pM \to \mathbb{R}$ at some point $p \in M$. For our purposes we give the following

Definition 2.1. Let $M \subset N$ be a smooth oriented isometric hypersurface immersion and let $\nu = e_0, \dots, e_n$ be a local adapted frame around $p \in M$. The bilinear mapping $h : T_pM \times T_pM \to \mathbb{R}$ given by the *Weingarten equations*

$$\begin{aligned} h_{ij} : &= h(e_i(p), e_j(p)) = \bar{g}(\bar{\nabla}_{e_i(p)}\nu(p), e_j(p)) \\ &= -\bar{g}(\nu(p), \bar{\nabla}_{e_i(p)}e_j(p)) \end{aligned} \tag{6}$$

is called *second fundamental form of M at p.*

The Weingarten equations are a valuable means to compute h_{ij}. However, in many situations it is preferable to work with local coordinates instead of adapted frames. Then, in order to carry out concrete computations, we need to know how (6) looks like in local coordinates.

Let therefore $x^1, \dots, x^n$ be local coordinates of M around p and let $y^0, \dots, y^n$ be local coordinates of N around $q := F(p)$. We compute h_{ij} in terms of the coordinate vector fields

$$\frac{\partial}{\partial x^i} \simeq DF\left(\frac{\partial}{\partial x^i}\right), \quad i = 1, \dots, n.$$

We also write $DF(\frac{\partial}{\partial x^i}) = \frac{\partial F}{\partial x^i}$. They are expressed in terms of the coordinate vector fields $\frac{\partial}{\partial y^\alpha}$, $\alpha = 0, \ldots, n$, of N by

$$\frac{\partial F}{\partial x^i} = \frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha},$$

where $(\frac{\partial F^\alpha}{\partial x^i})$ is the Jacobi matrix of F. Similarly we write $\nu = \nu^\alpha \frac{\partial}{\partial y^\alpha}$ for the unit normal. Observe that latin summation indices run from 1 to n, whereas greek indices run from 0 to n, a convention we agree to use from now on.

We now express the first identity of (6) in terms of the coordinate frame $\frac{\partial}{\partial x^i}$. It is

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}} \nu = \bar{\nabla}_{\left(\frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}\right)} \nu = \left(\frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} (\nu^\beta) + \frac{\partial F^\alpha}{\partial x^i} \nu^\tau \bar{\Gamma}^\beta_{\alpha\tau} \right) \frac{\partial}{\partial y^\beta} = \left(\frac{\partial \nu^\beta}{\partial x^i} + \frac{\partial F^\alpha}{\partial x^i} \nu^\tau \bar{\Gamma}^\beta_{\alpha\tau} \right) \frac{\partial}{\partial y^\beta}.$$

We thus get

$$h_{ij} = \bar{g} \left(\bar{\nabla}_{\frac{\partial}{\partial x^i}} \nu, \frac{\partial}{\partial x^j} \right) = \bar{g}_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^j} \left(\frac{\partial \nu^\beta}{\partial x^i} + \frac{\partial F^\alpha}{\partial x^i} \nu^\tau \bar{\Gamma}^\beta_{\alpha\tau} \right).$$

On the other hand we have

$$\bar{g}_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^j} g^{lm} \frac{\partial F^\beta}{\partial x^m} h_{li} = g^{lm} g_{jm} h_{li} = \delta^l_j h_{li} = h_{ij}.$$

These two relations yield

Lemma 2.2. *In terms of local coordinates the first version of the Weingarten equation* (6) *takes the form*

$$g^{lm} \frac{\partial F^\beta}{\partial x^m} h_{li} = \frac{\partial \nu^\beta}{\partial x^i} + \frac{\partial F^\alpha}{\partial x^i} \nu^\tau \bar{\Gamma}^\beta_{\alpha\tau}.$$

In order to compute the second identity in (6), we observe that the tangent space $T_q N$ can be decomposed into the direct sum of $T_q M$ and $(T_q M)^\perp = \text{span}(\nu)$. Thus it follows from $\bar{g}(\nu, \nu) \equiv 1$ that

$$\bar{g} \left(\nu, \bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \nu = \left(\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)^\perp,$$

where $^\perp$ denotes the orthogonal projection of a vector onto ν with respect to $\bar{g}$.

It is a well-known fact that the inner connection ∇ of M derived from the induced metric g can be recovered by projecting the ambient connection $\bar{\nabla}$ onto the tangent bundle of M along $\text{span}(\nu)$. We thus have

$$\left(\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)^\perp = \bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j},$$

whence we derive from (6)

$$\begin{aligned} -h_{ij}\nu &= \left(\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)^\perp = \bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \frac{\partial F^\alpha}{\partial x^i} \bar{\nabla}_{\frac{\partial}{\partial y^\alpha}} \left(\frac{\partial F^\beta}{\partial x^j} \frac{\partial}{\partial y^\beta} \right) - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \\ &= \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \bar{\nabla}_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta} + \frac{\partial F^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \left(\frac{\partial F^\beta}{\partial x^j} \right) \frac{\partial}{\partial y^\beta} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \\ &= \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} \bar{\Gamma}^\gamma_{\alpha\beta} \frac{\partial}{\partial y^\gamma} + \frac{\partial^2 F^\gamma}{\partial x^i \partial x^j} \frac{\partial}{\partial y^\gamma} - \Gamma^k_{ij} \frac{\partial F^\gamma}{\partial x^k} \frac{\partial}{\partial y^\gamma}. \end{aligned}$$

From $\nu = \nu^\gamma \frac{\partial}{\partial y^\gamma}$ it follows

Lemma 2.3. *In terms of local coordinates the second version of the Weingarten equation* (6) *takes the form*

$$-h_{ij}\nu^\gamma = \frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}\bar{\Gamma}^\gamma_{\alpha\beta} + \frac{\partial^2 F^\gamma}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^\gamma}{\partial x^k}.$$

These two lemmas will be of importance when we derive evolution equations of geometrical quantities.

For the second fundamental form h_{ij} one can establish the two crucial identities relating the geometries of M and N, the Gauss equation and the Codazzi equation. These relations are well-known and cited without proof.

Lemma 2.4. *Let $\nu = e_0, \ldots, e_n$ be an adapted frame around $p \in M \subset N$ as above. In terms of this frame the relation between the two curvature tensors R and $\bar{R}$ is given by the* Gauss equation

$$R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}.$$

In contrast to hypersurfaces in $\mathbb{R}^n$, the difference between two components of ∇h whose indices are cyclicly permuted, depends on the curvature of the ambient manifold. This is stated by the Codazzi equation.

Lemma 2.5. *In terms of the frame of lemma 2.4 we have the relation*

$$\nabla_k h_{ij} - \nabla_j h_{ik} = \bar{R}_{0ijk}.$$

Both equations are fundamental tools occurring at many different places throughout this text. In particular, an interesting consequence of the Codazzi equation is presented in the next section.

2.4 An identity for the Laplacian of h_{ij}

The Laplacian of a tensor T is defined by $\Delta T := g^{kl}\nabla_k\nabla_l T$. In particular, $\Delta h_{ij} := g^{kl}\nabla_k\nabla_l h_{ij}$. By means of the Codazzi equation, Simons (see [14]) derived an expression for Δh_{ij} which does not involve any covariant derivatives of h_{ij}. We are going to derive it in this section. Later on, this identity will be invoked in order to establish evolution equations.

A preliminary tool we need for the proof is to know how second covariant derivatives of h_{ij} commute.

Lemma 2.6. *Let $M \subset N$ be a smooth isometric hypersurface immersion as above. In terms of a local frame $e_1, \ldots, e_n \in TM$ around $p \in M$ the commutation of second covariant derivatives of h_{ij} is given by*

$$\nabla_k\nabla_l h_{ij} - \nabla_l\nabla_k h_{ij} = h_{mj}R_{lk}{}^m{}_i + h_{im}R_{lk}{}^m{}_j.$$

Proof. Since the identity is of tensorial nature, we may assume $\{e_i\}$ to be a normal coordinate frame. Then all Christoffel symbols vanish and we get

$$\nabla_k\nabla_l h_{ij} = e_k(e_l(h_{ij})) - e_k(\Gamma^m_{li})h_{mj} - e_k(\Gamma^m_{lj})h_{im},$$

as well as

$$\begin{aligned}\nabla_k\nabla_l h_{ij}-\nabla_l\nabla_k h_{ij} &= e_l(\Gamma^m_{ki})h_{mj}-e_k(\Gamma^m_{li})h_{mj}+e_l(\Gamma^m_{kj})h_{im}-e_k(\Gamma^m_{lj})h_{im}\\ &= h_{mj}R_{lk}{}^m{}_i+h_{im}R_{lk}{}^m{}_j,\end{aligned}$$

according to (5). □

Similar as above, for the following calculations it is handy to choose convenient coordinates to work with. This is possible since all considerations are pointwise. At $p\in M$ we choose local coordinates $x^1,\dots,x^n$ of M such that $g_{ij}=\delta_{ij}$ at p and we consider the frame $e_0,e_1,\dots,e_n$, where $e_0=\nu$ denotes a choice of the unit normal and $e_1=\frac{\partial}{\partial x^1},\dots,e_n=\frac{\partial}{\partial x^n}$.

Lemma 2.7. *In an arbitrary adapted frame we have the relation*

$$\bar\nabla_l\bar R_{0ijk}-\nabla_l\bar R_{0ijk}=-h_{lm}\bar R^m{}_{ijk}+h_{lj}\bar R_{0i0k}+h_{lk}\bar R_{0ij0}.$$

Proof. This is a tensorial identity and we may use the frame $e_o,\dots,e_n$ from above. Then $\bar\nabla_{e_i}e_0=\bar\Gamma^\alpha_{i0}e_\alpha$ and $g_{ij}=\delta_{ij}$, and it follows from the Weingarten equation

$$h_{ij}=\bar g(\bar\nabla_{e_i}\nu,e_j)=\bar g(\bar\nabla_{e_i}e_0,e_j)=\bar\Gamma^k_{i0}\delta_{kj}=\bar\Gamma^j_{i0}.$$

Moreover we derive from $|\nu|\equiv 1$

$$\bar\Gamma^0_{i0}=\bar g(\bar\nabla_{e_i}\nu,\nu)=\frac{1}{2}e_i(\bar g(\nu,\nu))=0.$$

Moreover we have $\bar\Gamma^k_{ij}=\Gamma^k_{ij}$ for $i,j,k\in\{1,\dots,n\}$, which can be seen by

$$\Gamma^k_{ij}e_k=\nabla_{e_i}e_j=(\bar\nabla_{e_i}e_j)^T=\bar\nabla_{e_i}e_j-\bar g(\bar\nabla_{e_i}e_j,\nu)\nu=\bar\Gamma^\alpha_{ij}e_\alpha-\bar\Gamma^0_{ij}e_0=\bar\Gamma^k_{ij}e_k.$$

Now we have to distinguish two ways of thinking of $\bar R_{0ijk}$. On the one hand it is a component of the $(4,0)$-tensor $\bar R$ on N. The covariant derivative $\bar\nabla$ of $\bar R$ as a $(4,0)$-tensor then looks like

$$\bar\nabla_l\bar R_{0ijk}=e_l(\bar R_{0ijk})-\bar\Gamma^\xi_{l0}\bar R_{\xi ijk}-\bar\Gamma^\xi_{li}\bar R_{0\xi jk}-\bar\Gamma^\xi_{lj}\bar R_{0i\xi k}-\bar\Gamma^\xi_{lk}\bar R_{0ij\xi},$$

where we already have inserted the correct indices. On the other hand, restricted on M, $\bar R_{0ijk}$ is a $(3,0)$-tensor in the last three indices with the covariant derivative

$$\nabla_l\bar R_{0ijk}=e_l(\bar R_{0ijk})-\Gamma^m_{li}\bar R_{0mjk}-\Gamma^m_{lj}\bar R_{0imk}-\Gamma^m_{lk}\bar R_{0ijm}.$$

Thus

$$\begin{aligned}\bar\nabla_l\bar R_{0ijk}-\nabla_l\bar R_{0ijk} &= -\bar\Gamma^\xi_{l0}\bar R_{\xi ijk}-\bar\Gamma^0_{li}\bar R_{00jk}-\bar\Gamma^0_{lj}\bar R_{0i0k}-\bar\Gamma^0_{lk}\bar R_{0ij0}\\ &= -h_{lm}R^m{}_{ijk}+h_{lj}\bar R_{0i0k}+h_{lk}\bar R_{0ij0}\end{aligned}$$

since $h_{ij}=-\bar\Gamma^0_{ij}$. □

Let us now consider the term $\nabla_k\nabla_l h_{ij}$. Our goal is to gradually swap the first pair of indices with the second. We will proceed in three steps: Applying the Codazzi equation (lemma 2.5) we get

$$\nabla_k\nabla_l h_{ij}=\nabla_k\nabla_j h_{il}+\nabla_k\bar R_{0ijl}.$$

The next step is to commute the second covariant derivatives by lemma 2.6, which yields

$$\nabla_k \nabla_l h_{ij} = \nabla_j \nabla_k h_{il} + h_{ml} R_{jk}{}^{m}{}_{i} + h_{im} R_{jk}{}^{m}{}_{l} + \nabla_k \bar{R}_{0ijl}.$$

Finally we invoke the Codazzi equation once more, whence

$$\nabla_k \nabla_l h_{ij} = \nabla_j \nabla_i h_{lk} + \nabla_j \bar{R}_{0lik} + \nabla_k \bar{R}_{0ijl} + h_{ml} R_{jk}{}^{m}{}_{i} + h_{im} R_{jk}{}^{m}{}_{l}.$$

Notice that the second and the third term of the right hand side are $\nabla \bar{R}$-terms which can be turned into $\bar{\nabla}\bar{R}$-terms by lemma 2.7. Moreover, the R-terms occurring in the two last terms can be transformed into $\bar{R}$-terms by lemma 2.4. Altogether this yields

$$\begin{aligned} \nabla_k \nabla_l h_{ij} \quad = \quad & \nabla_j \nabla_i h_{lk} + \bar{\nabla}_j \bar{R}_{0lik} + \bar{\nabla}_k \bar{R}_{0ijl} \\ & - h_{ij} \bar{R}_{0l0k} - h_{kj} \bar{R}_{0li0} + h_{mj} R^{m}{}_{lik} \\ & - h_{jk} \bar{R}_{0i0l} - h_{lk} \bar{R}_{0ij0} + h_{mk} R^{m}{}_{ijl} \\ & + h_{lm} R^{m}{}_{ijk} + h_{mi} R^{m}{}_{ljk} \\ & + h_{lm} h^{m}_{j} h_{ik} - h_{lm} h^{m}_{k} h_{ij} + h_{mi} h^{m}_{j} h_{lk} - h_{mi} h^{m}_{k} h_{lj}. \end{aligned} \tag{7}$$

Remark 2.8. For tensors S and T of the same type on M we denote the scalar product with respect to g by $\langle S, T \rangle$. For example let S and T be of type $(2,0)$ with components S_{ij} and T_{ij}, respectively. Then $\langle S, T \rangle = \langle S_{ij}, T_{ij} \rangle = g^{ik} g^{jl} S_{ij} T_{kl}$. Furthermore, $|S_{ij}|^2 = \langle S_{ij}, S_{ij} \rangle$ and, in particular, the total curvature is denoted by $|A|^2 = |h_{ij}|^2$.

Now it is straightforward to prove the following

Lemma 2.9. *Let $M \subset N$ be an isometric hypersurface immersion of Riemannian manifolds. The Laplacian of the second fundamental form then satisfies the identity*

$$\begin{aligned} \triangle h_{ij} \quad = \quad & \nabla_i \nabla_j H + H h_{il} h_{lj} - |A|^2 h_{ij} + H \bar{R}_{0i0j} \\ & - h_{ij} \bar{R}_{0l0}{}^{l} + h_{jl} \bar{R}^{l}{}_{mi}{}^{m} + h_{il} \bar{R}^{l}{}_{mj}{}^{m} - 2 h_{lm} \bar{R}^{l}{}_{i}{}^{m}{}_{j} \\ & + \bar{\nabla}_j \bar{R}_{0li}{}^{l} + \bar{\nabla}_l \bar{R}_{0ij}{}^{l} \end{aligned} \tag{8}$$

with respect to an arbitrary adapted frame $\nu = e_0, e_1, \ldots, e_n$ of M.

Proof. This equation is simply the contraction of (7) by g^{lk}. Note that g and the covariant derivate can be interchanged arbitrarily according to the fact that $\bar{g}$ is parallel, i.e. $\bar{\nabla}\bar{g} = 0$. □

Remark 2.10. Note that the above identity recoveres Δh_{ij} of the immersed hypersurface M by means of h_{ij}, $\bar{R}$, $\nabla \bar{R}$ and the second derivative of H. It simplifies considerably in Euclidean space as all the terms involving ambient curvature vanish.

We will later need a direct consequence of (8):

Corollary 2.11. *We have*

$$\begin{aligned} \frac{1}{2} \Delta |A|^2 \quad = \quad & \langle h_{ij}, \nabla_i \nabla_j H \rangle + |\nabla_k h_{ij}|^2 - |A|^4 + H h^{ij} h_{il} h^{l}_{j} \\ + \quad & H h^{ij} \bar{R}_{0i0j} - |A|^2 \bar{R}_{0l0}{}^{l} + 2 h^{ij} h_{jl} \bar{R}^{l}{}_{mi}{}^{m} - 2 h^{ij} h_{lm} \bar{R}^{l}{}_{i}{}^{m}{}_{j} \\ + \quad & h^{ij} (\bar{\nabla}_j \bar{R}_{0li}{}^{l} + \bar{\nabla}_l \bar{R}_{0ij}{}^{l}). \end{aligned} \tag{9}$$

Proof. A straightforward calculation yields

$$\Delta|A|^2 = 2h^{ij}\Delta h_{ij} + 2|\nabla_k h_{ij}|^2.$$

This and the inner multiplication of (8) by h_{ij} proves the assertion. □

Lemma 2.9 and its corollary are essential relations we will need later. We are now ready to introduce the main topic all our considerations are based upon: the mean curvature flow.

3 Mean curvature flow and geometric evolution equations

Roughly speaking, by *mean curvature flow* one denotes the motion of an isometrically immersed hypersurface $M_0 \subset N$ along its unit normal field with the velocity varying from point to point, equalling its mean curvature at each point. This is an initial value problem, whose solution - if existing - depends on the nature of M_0. The motion of the surface, in turn, impacts the evolution of the mean curvature as well as other geometrical quantities according to the corresponding *geometric evolution equations.* As already mentioned in the introduction, one usually starts with an initial hypersurface and considers its behaviour under mean curvature flow.

In the present chapter we first introduce the notion of mean curvature flow. Then we explain how it can be rewritten as a quasilinear parabolic system of partial differential equations and briefly discuss the (short-time) existence of solutions. As direct implications we afterwards derive evolution equations for several geometrical quantities, especially aiming to the evolution of the second fundamental form.

3.1 Mean curvature flow

Let $n \geq 2$. We start by considering a smooth *initial* hypersurface M_0 of dimension n which is isometrically immersed in an $(n+1)$-dimensional Riemannian manifold N by the mapping

$$F_0 : M \to N.$$

Suppose that F_0 may be extended to a mapping

$$F : M \times [0, T[\to N, \quad 0 < T \leq \infty,$$

with $F(\cdot, 0) = F_0$. Suppose further that each $F(\cdot, t)$, $0 \leq t \leq T$ be an isometric immersion of M. Consequently, the metric g and all other geometric quantities of M depend on t. If we consider t as a time parameter, we interpret the family $\{M_t = F(\cdot, t)(M), t \in [0, T[\}$ as the *evolution* of the initial surface M_0 during the time interval $[0, T]$.

The evolution of M_0 can be subjected to certain rules of motion. Let $t \in [0, T[$ and $p \in M_t$. Let $\nu(\cdot, t)$ be a local unit normal field of M_t around p. We then define:

Definition 3.1. The mapping F defined above describes an evolution of the initial hypersurface M_0 according to the *mean curvature flow* if it satisfies the evolution equation

$$\frac{\partial}{\partial t} F(p, t) = -H(p, t)\nu(p, t) \quad \text{for all } p \in M \text{ and } t \in [0, T[\tag{10}$$

with the initial condition

$$F(\cdot, 0) = F_0.$$

Here $H(p, t)$ denotes the *mean curvature* $H = g^{ij} h_{ij}$ at (p, t).

Remark 3.2. The negative sign in (10) stems from the fact that for oriented closed surfaces without boundary and with strictly positive mean curvature such as the sphere we usually choose ν to be the outer unit normal. In this case the negative sign makes the surface shrink and not expand.

We need a reformulation of (10) to concretely work with. For this purpose we choose local coordinates $y = (y^0, \dots, y^n)$ of N around $F(p,t)$ and local coordinates $x = (x^1, \dots, x^n)$ around (p,t). Suppose that $\frac{\partial F^\alpha}{\partial t}$ and ν^α denote the respective coefficients of the local representations of $DF(\frac{\partial}{\partial t}) = \frac{\partial F}{\partial t}$ and ν, given by the relations

$$\nu = \nu^\alpha \frac{\partial}{\partial y^\alpha} \quad \text{and} \quad DF\left(\frac{\partial}{\partial t}\right) = \frac{\partial F^\alpha}{\partial t}\frac{\partial}{\partial y^\alpha} \quad \text{at } (p,t).$$

Then (10) decomposes into the $n+1$ components

$$\frac{\partial F^\alpha}{\partial t} = -H\nu^\alpha \quad \text{for } \alpha = 1, \dots, n+1. \tag{11}$$

This local version, in turn, allows to be rewritten by means of lemma 2.3, such that in local coordinates it follows from (11) that we have at (p,t)

$$\begin{aligned} \frac{\partial F^\alpha}{\partial t} &= -g^{ij}h_{ij}\nu^\alpha = g^{ij}\left(\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\bar{\Gamma}^\alpha_{\beta\gamma} + \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k}\right) \\ &= \Delta_t F^\alpha + g^{ij}\left(\frac{\partial F^\beta}{\partial x^i}\frac{\partial F^\gamma}{\partial x^j}\bar{\Gamma}^\alpha_{\beta\gamma}\right), \end{aligned} \tag{12}$$

where

$$\Delta_t F^\alpha := g^{ij}\nabla_i\nabla_j F^\alpha = g^{ij}\left(\frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial F^\alpha}{\partial x^k}\right).$$

It is worth noting that the F^α are local real-valued functions $M \times [0,T[\to \mathbb{R}$ around (p,t) defined by

$$F^\alpha(p,t)e_\alpha = y(F(p,t)).$$

Here $\{e_\alpha\}$ denotes the standard basis of $\mathbb{R}^{n+1}$, whereas y is the local diffeomorphism $N \to \mathbb{R}^{n+1}$ corresponding to the coordinate frame $\{\frac{\partial}{\partial y^\alpha}\}$.

Thus (12) shows that locally the mean curvature flow equation takes the form of a parabolic system of second order, which is quasilinear since it is linear with coefficients depending on F^α and $\frac{\partial F^\alpha}{\partial x^i}$.

Up to now we have assumed that F satisfy (10) for a given initial surface M_0. This, of course, does not imply that such a mapping F actually exists for any given M_0. However, for the case that M_0 is smooth, we have the following theorem which we cite from [8].

Theorem 3.3. *For any given smooth initial hypersurface M_0 which is isometrically immersed in an ambient Riemannian manifold N we have a smooth solution F satisfying* (10) *at least on some short time interval* $[0,T[$, $0 < T \leq \infty$.

Remark 3.4. We refer to [6] for further details. Note that the linearization of (12) reveals that the system is only weakly parabolic. This is due to solutions which do not actually move the surface while only describing a family of different parametrizations of the same surface (tangential movements). One therefore has to decompose all possible solutions into their tangential and normal movements and then comprise all those with the same normal movement in one equivalence class. Concerning this quotient space of solutions the equation gets strictly parabolic, which finally implies the short-time existence.

From now on let us assume the initial surface M_0 to be smooth. We then denote by $[0,T[$, $T \leq \infty$, the maximum time interval on which the mean curvature flow solution exists.

The mean curvature flow is reflected by the change of the geometric quantities of M_t. Their evolution, in turn, can be derived by means of the mean curvature flow equation, which is the purpose of the next section.

3.2 Evolution equations of some geometric quantities

Our aim is basically to understand how the geometry of the initial surface M_0 changes while moving by mean curvature flow. We are interested in any kind of information about both intrinsic quantities such as the metric and extrinsic quantities such as the second fundamental form, the mean curvature or the unit normal.

In particular we will consider how these quantities *change locally* while the flow goes on, in order to draw conclusions about the *global* development of the shape of the moving hypersurface. This is the reason why it is necessary to derive evolution equations, or 'time derivatives', for the quantities in question. Since the mean curvature flow equation describes the evolution of M_t (and thus its intrinsic geometry) by means of extrinsic quantities (mean curvature and unit normal), it accordingly will turn out that the time derivative of intrinsic quantities will depend on extrinsic quantities as well.

Evolution equations are manifestations of the mean curvature flow equation in many different situations. They provide the basic key to successfully examine the properties of the moving surface. As the various evolution equations inherit the parabolic nature of the mean curvature flow equation, one hopes to be able to apply techniques from the theory of parabolic PDE, which will be of crucial importance in chapter 4. In the present section we mainly derive the evolution equation of the metric and the unit normal, whereas the following section deals with the evolution of the second fundamental form h_{ij}, implying several evolution equations of quantities related to h_{ij}.

Suppose as above that we have a maximal solution $F : M \times [0, T[\to N$ for the mean curvature flow of a given smooth initial surface $M_0 \subset N$, yielding the family of hypersurfaces $\{M_t, t \in [0, T[\}$.

Observe that for a chart (U, x) of M around $p \in M$ we have the coordinate vector fields

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \quad \text{and} \quad \frac{\partial}{\partial t}$$

at our disposal. The $\frac{\partial}{\partial x^i}$ may be identified with their images under the differential DF due to the fact that $F(\cdot, t)$ is an isometry for each $t \in [0, T[$. However, it is important to understand how the image of $\frac{\partial}{\partial t}$ should be interpreted. Observe that for a real-valued function $f \in C^\infty(N)$ it is determined by the fact that F satisfies the mean curvature equation, that is

$$DF\left(\frac{\partial}{\partial t}\right)(f) = -HD_\nu f. \tag{13}$$

In other words, if we want to know how f changes in time at a fixed point $(p, t) \in M \times [0, T[$, we switch to the immersion point of view, take $F(p, t) \in N$ instead and consider the change of f in direction of $-H\nu$. The same holds for smooth functions f on M, provided that $H \neq 0$. Then f can be considered as a smooth function on N at least in a small time neighbourhood.

Of course the same works for smooth tensor fields on N. As an example we get $\frac{\partial}{\partial t}\bar{g} = 0$, which is due to $\bar{\nabla}\bar{g} = 0$ and

$$\frac{\partial}{\partial t}\bar{g} = -H\bar{\nabla}_\nu \bar{g} = -H(\bar{\nabla}\bar{g})(\nu) = 0.$$

We will use this fact in the following calculations without mentioning.

We now consider the metric.

Lemma 3.5. *Let $(p,t) \in M \times [0,T[$. The evolution of the metric g of M_t at p is given by*

$$\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}.$$

Proof. Since this identity is of tensorial nature, we may arbitrarily choose the coordinates we use for the calculation - both for M_t and N.

Around p we therefore choose coordinates $x^1, \ldots, x^n$ such that at $q := F(p,t)$ the induced metric is orthonormal, i.e.

$$g_{ij} = \bar{g}\left(\frac{\partial F}{\partial x^i}(q), \frac{\partial F}{\partial x^j}(q)\right) = \delta_{ij}.$$

Moreover, by starting geodesics at p in the directions $\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}$ and ν we obtain normal coordinates y^α such that $\nu^\alpha = -\delta_0^\alpha$ and $\frac{\partial F^\alpha}{\partial x^i} = \delta_i^\alpha$ at $F(p,t)$. Then all $\bar{\Gamma}_{\alpha\beta}^\gamma$ vanish at q and the Weingarten equation of lemma 2.3 takes the form

$$\frac{\partial \nu^\beta}{\partial x^i} = h_{li} \frac{\partial F^\beta}{\partial x^i}. \tag{14}$$

Note that tangential vectors of M_t may be thought of in two ways. On the one hand they are derivations operating on smooth functions defined on M. On the other hand, in view of the fact that M_t is immersed in N, they can be locally extended to derivations operating on smooth functions on N. Then they may be represented as linear combinations of tangential vectors of N, whose coefficients are smooth functions on M. These two points of view both occur in the calculations below.

Due to $\bar{\Gamma}_{ij}^k = 0$ at $F(p,t)$, we have at (p,t)

$$\frac{\partial}{\partial t} g_{ij} = \frac{\partial}{\partial t} \bar{g}\left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) = \bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) + \bar{g}\left(\frac{\partial F}{\partial x^i}, \bar{\nabla}_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial x^j}\right).$$

Moreover,

$$\begin{aligned}
\bar{g}\left(\bar{\nabla}_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) &= \bar{g}\left(\frac{\partial F}{\partial t}\left(\frac{\partial F^\beta}{\partial x^i}\right)\frac{\partial}{\partial y^\beta}, \frac{\partial F}{\partial x^j}\right) = \bar{g}\left(\frac{\partial F}{\partial x^i}\left(\frac{\partial F^\beta}{\partial t}\right)\frac{\partial}{\partial y^\beta}, \frac{\partial F}{\partial x^j}\right) \\
= \; & \bar{g}\left(\frac{\partial F}{\partial x^i}\left(-H\nu^\beta\right)\frac{\partial}{\partial y^\beta}, \frac{\partial F}{\partial x^j}\right) = -H\bar{g}\left(\frac{\partial \nu^\beta}{\partial x^i}\frac{\partial}{\partial y^\beta}, \frac{\partial F}{\partial x^j}\right) - \bar{g}\left(\nu\frac{\partial H}{\partial x^i}, \frac{\partial F}{\partial x^j}\right) \\
= \; & -H\bar{g}\left(h_{il}\frac{\partial F^\beta}{\partial x^l}\frac{\partial}{\partial y^\beta}, \frac{\partial F}{\partial x^j}\right) = -Hh_{il}\delta_j^l.
\end{aligned}$$

Note that in the second last identity we took advantage of the special form of our coordinates as well as the Weingarten equation (14).

A similar calculation for the second summand yields $\frac{\partial}{\partial t} g_{ij} = -Hh_{il}\delta_j^l - Hh_{jl}\delta_i^l = -2Hh_{ij}$. □

Another object of interest is the unit normal ν.

Lemma 3.6. *Let $(p,t) \in M \times [0,T[$. The evolution of the unit normal ν of M_t at p is given by*

$$\frac{\partial}{\partial t}\nu = \nabla H.$$

Proof. First observe that $\frac{\partial}{\partial t}\nu$ and ν are orthogonal due to

$$\bar{g}\left(\frac{\partial}{\partial t}\nu, \nu\right) = \frac{1}{2}\frac{\partial}{\partial t}\bar{g}(\nu, \nu) = 0,$$

since $\bar{g}(\nu, \nu) \equiv 1$. Thus $\frac{\partial}{\partial t}\nu \in T_{F(t,p)}M_t$. Then, if we choose coordinates at p such that $g_{ij} = \delta_{ij}$, due to a well-known fact from linear algebra we have

$$\frac{\partial}{\partial t}\nu = \sum_i \bar{g}\left(\frac{\partial}{\partial t}\nu, \frac{\partial F}{\partial x^i}\right)\frac{\partial F}{\partial x^i}.$$

Since ν and $\frac{\partial F}{\partial x^i}$ are orthogonal, we obtain by the product rule that the above term is equal to

$$-\sum_i \bar{g}\left(\nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x^i}\right)\frac{\partial F}{\partial x^i} = \sum_i \left(\nu, \frac{\partial}{\partial x^i}(H\nu)\right)\frac{\partial F}{\partial x^i} = \sum_i \frac{\partial}{\partial x^i}H\frac{\partial F}{\partial x^i} = \nabla H.$$

In the second equality we once more have invoked the mean curvature flow equality. □

With this knowledge about the behaviour of g and ν, we are now ready to derive the evolution equation of the quantity we are mainly interested in, the second fundamental form.

3.3 The evolution of the second fundamental form

As we will see, convexity is a geometrical property directly related to the second fundamental form. In order to study the behaviour of convexity under mean curvature flow it is therefore of special interest to know how the second fundamental form evolves.

Again the idea consists of simplifying the calculations by choosing normal coordinates $y^0, \ldots, y^n$ around a fixed point $q = F(t, p) \in N$. We are allowed to do that since we carry out the calculations only at q and since it will turn out that suitable manipulations will reduce all terms involved to tensorial expressions.

We then look at the definition of h_{ij} given by the second version of the Weingarten equation 2.3. If we inner multiply 2.3 by ν with respect to $\bar{g}$, we get

$$-h_{ij} = \bar{g}\left(\frac{\partial^2 F}{\partial x^i \partial x^j}, \nu\right) + \bar{g}_{\alpha\beta}\nu^\alpha \bar{\Gamma}^\beta_{\gamma\delta}\frac{\partial F^\gamma}{\partial x^i}\frac{\partial F^\delta}{\partial x^j}, \tag{15}$$

since $\bar{g}(\nu, \nu) \equiv 1$ and $\bar{g}(\frac{\partial F}{\partial x^k}, \nu) = 0$.

If we differentiate the second term of the right hand side by the product rule, due to our choice of normal coordinates the only remaining term will be the one involving the derivative of $\bar{\Gamma}^\beta_{\gamma\delta}$. We get

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= -\frac{\partial}{\partial t}\left(\frac{\partial^2 F}{\partial x^i \partial x^j}, \nu\right) - \bar{g}_{\alpha\beta}\nu^\alpha \frac{\partial}{\partial t}\left(\bar{\Gamma}^\beta_{\gamma\delta}\right)\frac{\partial F^\gamma}{\partial x^i}\frac{\partial F^\delta}{\partial x^j} \\ &= -\frac{\partial}{\partial t}\left(\frac{\partial^2 F}{\partial x^i \partial x^j}, \nu\right) + H\bar{g}_{\alpha\beta}\nu^\alpha\nu^\tau \frac{\partial}{\partial y^\tau}\left(\bar{\Gamma}^\beta_{\gamma\delta}\right)\frac{\partial F^\gamma}{\partial x^i}\frac{\partial F^\delta}{\partial x^j},\end{aligned} \tag{16}$$

where the second identity follows from equation (13).

By the product rule for inner products, equation (13) and lemma 3.6 along with the expression of the gradient given by

$$\nabla H = \frac{\partial}{\partial x^i} H g^{ij} \frac{\partial F}{\partial x^j},$$

we obtain from (16) the relation

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} &= \bar{g}\left(\frac{\partial^2}{\partial x^i \partial x^j}(H\nu), \nu\right) - \bar{g}\left(\frac{\partial^2 F}{\partial x^i \partial x^j}, \frac{\partial}{\partial x^l} H g^{lm} \frac{\partial F}{\partial x^m}\right) \\ &\quad + H \bar{g}_{\alpha\beta} \nu^\tau \frac{\partial F^\gamma}{\partial x^i} \nu^\alpha \frac{\partial F^\delta}{\partial x^j} \frac{\partial}{\partial y^\tau}\left(\bar{\Gamma}^\beta_{\gamma\delta}\right). \end{aligned} \tag{17}$$

Furthermore, by $\bar{g}\left(\nabla_{\frac{\partial}{\partial x^i}} \nu, \nu\right) = \frac{1}{2}\frac{\partial}{\partial x^i}\bar{g}(\nu,\nu) = 0$ and the product rule we derive for the first term of the right hand side of the above identity

$$\bar{g}\left(\frac{\partial^2}{\partial x^i \partial x^j}(H\nu), \nu\right) = \frac{\partial^2}{\partial x^i \partial x^j} H + H\bar{g}\left(\frac{\partial^2}{\partial x^i \partial x^j}\nu, \nu\right), \tag{18}$$

whereas it follows from the Weingarten equations, the product rule and $\bar{\Gamma}^\gamma_{\alpha\beta} = 0$ that

$$\begin{aligned} H\bar{g}\left(\frac{\partial^2}{\partial x^i \partial x^j}\nu, \nu\right) &= H\bar{g}\left(\frac{\partial}{\partial x^i}\left(h_{jl} g^{lm} \frac{\partial F}{\partial x^m}\right) - \frac{\partial}{\partial x^i}\left(\bar{\Gamma}^\beta_{\delta\tau} \frac{\partial F^\delta}{\partial x^j} \nu^\tau \frac{\partial}{\partial y^\beta}\right), \nu\right) \\ &= H\bar{g}\left(h_{jl} g^{lm} \frac{\partial^2 F}{\partial x^i \partial x^m}, \nu\right) - H\bar{g}_{\alpha\beta} \frac{\partial F^\gamma}{\partial x^i} \frac{\partial}{\partial y^\gamma}\left(\bar{\Gamma}^\beta_{\delta\tau}\right) \frac{\partial F^\delta}{\partial x^j} \nu^\tau \nu^\alpha \\ &= -H h_j^m h_{im} - H\bar{g}_{\alpha\beta} \frac{\partial F^\gamma}{\partial x^i} \frac{\partial}{\partial y^\gamma}\left(\bar{\Gamma}^\beta_{\tau\delta}\right) \frac{\partial F^\delta}{\partial x^j} \nu^\tau \nu^\alpha. \end{aligned} \tag{19}$$

Note that in the first identity we have used the first version of the Weingarten equations (lemma 2.2) and in the last equation we have employed the second version (lemma 2.3). In both relations we also have taken advantage of the fact that ν is orthogonal to $\frac{\partial F}{\partial x^i}$.

On the other hand, we compute for the second term of the right hand side of (17) by means of lemma (2.3)

$$\begin{aligned} \bar{g}\left(\frac{\partial^2 F}{\partial x^i \partial x^j}, \frac{\partial}{\partial x^l} H g^{lm} \frac{\partial F}{\partial x^m}\right) &= \bar{g}\left(\Gamma^k_{ij} \frac{\partial F}{\partial x^k} - h_{ij}\nu, \frac{\partial}{\partial x^l} H g^{lm} \frac{\partial F}{\partial x^m}\right) \\ = \bar{g}\left(\Gamma^k_{ij} \frac{\partial F}{\partial x^k}, \frac{\partial}{\partial x^l} H g^{lm} \frac{\partial F}{\partial x^m}\right) &= \Gamma^k_{ij} g_{km} \frac{\partial}{\partial x^l} H g^{lm} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} H. \end{aligned} \tag{20}$$

Altogether it follows from (17), (18), (19), and (20)

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} &= \frac{\partial^2}{\partial x^i \partial x^j} H - \Gamma^k_{ij} \frac{\partial}{\partial x^k} H - H h_j^m h_{im} \\ &\quad + H\bar{g}_{\alpha\beta} \nu^\tau \frac{\partial F^\gamma}{\partial x^i} \nu^\alpha \frac{\partial F^\delta}{\partial x^j} \left[\frac{\partial}{\partial y^\tau}\left(\bar{\Gamma}^\beta_{\gamma\delta}\right) - \frac{\partial}{\partial y^\gamma}\left(\bar{\Gamma}^\beta_{\tau\delta}\right)\right] \\ &= \nabla_i \nabla_j H - H h_j^m h_{im} + H\bar{g}_{\alpha\beta} \nu^\tau \frac{\partial F^\gamma}{\partial x^i} \nu^\alpha \frac{\partial F^\delta}{\partial x^j} \bar{R}_{\tau\gamma}{}^\beta{}_\delta \\ &= \nabla_i \nabla_j H - H h_j^m h_{im} + H \bar{R}_{0i0j}. \end{aligned} \tag{21}$$

Note that in the second last step we have invoked the definition (4) of $\bar{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}$ simplified by use of normal coordinates.

This is already a short, nice evolution equation. We may, however, improve its structure towards the parabolic shape we want it to take. For this purpose observe that the terms of the right hand side occur in the identity (8) for Δh_{ij}, which we have derived in chapter 2.4. By taking advantage of this identity we get the following

Theorem 3.7. *Let $(p,t) \in M \times [0,T[$. The evolution of the second fundamental form h_{ij} at p is given by*

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij} &= \Delta h_{ij} - 2H h_j^m h_{im} + |A|^2 h_{ij} + h_{ij} \bar{R}_{0l0}{}^{l} \\ &\quad - h_{jl} \bar{R}^{l}{}_{mi}{}^{m} - h_{il} \bar{R}^{l}{}_{mj}{}^{m} + 2h_{lm} \bar{R}^{l}{}_{i}{}^{m}{}_{j} - \bar{\nabla}_j \bar{R}_{0li}{}^{l} - \bar{\nabla}_l \bar{R}_{0ij}{}^{l}\end{aligned}$$

with respect to an arbitrary adapted frame $\nu = e_0, \ldots, e_n$ of M_t around $F(p,t)$.

Remark 3.8. The parabolic structure of this tensorial evolution equation is particularly useful, since we will be able to adapt techniques from the theory of "ordinary" parabolic partial differential equations such as the maximum principle in order to to give statements about its global behaviour.

Remark 3.9. It is important to realize which quantities contribute to the way h_{ij} changes: Apart from the Laplacian of h_{ij} there are terms involving the ambient curvature $\bar{R}$ as well as gradients of $\bar{R}$. Thus, as we will see in the next chapter, both $\bar{R}$ and its gradient might work against the preservation of convexity. As we already have announced it is inevitable to control both quantities in order to preserve convexity under mean curvature flow in general Riemannian manifolds.

As a straightforward consequence of the above theorem we get the evolution equation for the mean curvature H by simply contracting (21).

Corollary 3.10. *The mean curvature satisfies the evolution equation*

$$\frac{\partial}{\partial t} H = \Delta H + H(|A|^2 + \bar{R}_{0l0}{}^{l}).$$

Proof. The result follows from (21) and the relation

$$\frac{\partial}{\partial t} H = h_{ij} \frac{\partial}{\partial t} g^{ij} + g^{ij} \frac{\partial}{\partial t} h_{ij} = 2H|A|^2 + g^{ij} \frac{\partial}{\partial t} h_{ij},$$

where in the last equality we have used

$$\frac{\partial}{\partial t} g^{ij} = -g^{il} g^{jk} \frac{\partial}{\partial t} g_{lk} = 2H h^{ij}.$$

□

This is a parapolic PDE, too. As the movement is governed by mean curvature, this equation is of special importance. For instance, the Ricci curvature term $\bar{R}_{0l0}{}^{l}$ directly impacts the changing rate of H and thus the flow. It will be this very equation which we exploit in the next chapter to gain more information about the maximum time interval $[=, T[$ of the flow.

Another consequence we need later is the evolution of the total curvature $|A|^2 = h_i^j h_j^i$. We get this by using the product rule.

Corollary 3.11. *The total curvature $|A|^2$ evolves according to*

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{R}_{0l0}{}^{l}) - 4(h^{ij}h_j^m\bar{R}_{mli}{}^{l} - h^{ij}h^{lm}\bar{R}_{milj}) \\ &\quad -2h_{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^{l} + \bar{\nabla}_l\bar{R}_{0ij}{}^{l}).\end{aligned}$$

We are now well prepared to turn our interest to convex initial surfaces M_0. In particular, the next chapter shows under which circumstances it can be guaranteed that initial convexity - in an appropriately broader sense - is preserved during the flow.

4 Evolving hypersurfaces and convexity

In this chapter we present the main theorem of this paper. It basically states that a suitably broadened notion of convexity of a closed (i.e. compact and boundary-free) initial surface $M_0 \subset N$ is preserved under mean curvature flow, provided that the curvature of N and its gradient are suitably bounded. This is one of the central results of [8]. During its proof, the interplay between the geometric properties of the ambient space N and the evolving hypersurface M_t will get clear.

At this stage one of the important features of mean curvature flow can be exploited - its parabolic structure and thus the possibility of applying maximum principles. Before proving the main theorem, we employ the parabolic maximum principle to demonstrate that for such closed initial surfaces with H sufficiently large a bounded solution of mean curvature flow can only exist on a finite time interval $[0, T[$, $T < \infty$, if the sectional curvature of N is bounded from below. Subsequently we extend the maximum principle to symmetric tensor fields with the aim of applying it to the evolution equation of the second fundamental form. This extension of the maximum principle is at the very core of the convexity theorem, and we will gradually build the proof around it, revealing the connections step by step.

We begin with the definition of convexity.

Let $M \subset N$ be an isometric hypersurface immersion. Furthermore let h_{ij} be the second fundamental form of M. Note that since h_{ij} is a symmetric tensor, there is always an orthonormal basis $\{v_i\}$ of T_pM in terms of which $h_{ij} = \operatorname{diag}(\kappa_1, \ldots, \kappa_n)$, where κ_i is called the ith principal curvature of M at p. We define:

Definition 4.1. A hypersurface $M \subset N$ is called *convex*, if all principal curvatures κ_i are nonnegative on the whole of M, or, equivalently, if h_{ij} is positive semidefinite everywhere on M. We then agree to write $h_{ij} \geq 0$ if $h_{ij}v^iv^j \geq 0$ for every vector $v = \{v^i\}$. Similarly, convexity is said to be *strict*, if $h_{ij} > 0$.

Remark 4.2. Sometimes the above property is referred to as *locally convex*. We will, however, keep the above definition, since confusions are ruled out.

From now on we will restrict our considerations to *closed smooth initial hypersurfaces* $M_0 \subset N$. Recall that N is an arbitrary Riemannian manifold of dimension $n + 1$, $n \geq 2$. Recall also that we denote by $[0, T[$, $T \leq \infty$, the maximum time interval on which the solution exists.

4.1 Maximum principle and mean curvature

The common parabolic maximum principle in $\mathbb{R}^n$ holds on manifolds as well. We cite a special version.

Lemma 4.3. *Let M be a compact n-dimensional Riemannian manifold without boundary. Let $T \leq \infty$ and $f : M \times [0, T] \to \mathbb{R}$ be a smooth function with*

$$\frac{\partial}{\partial t} f \leq \Delta f + b^i \frac{\partial}{\partial x^i} f, \quad b^i \in C^\infty(M \times [0, T]).$$

Then $\max_{M \times [0,T]} f \leq \max_{M \times \{0\}} f$.

Likewise, if

$$\frac{\partial}{\partial t} f \geq \Delta f + b^i \frac{\partial}{\partial x^i} f, \quad b^i \in C^\infty(M \times [0,T]),$$

we have $\min\limits_{M\times[0,T]} f \geq \min\limits_{M\times\{0\}} f$.

Remark 4.4. In particular we will need the second statement of this lemma.

We want to apply the maximum principle to the evolution equation for the mean curvature

$$\frac{\partial}{\partial t} H = \Delta H + H(|A|^2 + \bar{R}_{0l0}{}^l)$$

in order to show that the minimum of H is attained on the initial hypersurface M_0. Let us assume $H \geq 0$ on M_0.

We first note that $|A|^2 \geq H^2/n$, whence we have

$$\frac{\partial}{\partial t} H \geq \Delta H + H(\frac{1}{n} H^2 + \bar{R}_{0l0}{}^l). \tag{22}$$

Clearly, the second part of the above lemma immediately applies if we demand that the term $H(\frac{1}{n}H^2 + \bar{R}_{0l0}{}^l)$ be nonnegative.

We now consequently impose the first obstructions upon M_0 and the geometry of N. Let us assume that the sectional curvature of N be bounded from below by $-K_1$ everywhere in N, $K_1 \geq 0$. Then it follows $\bar{R}_{0l0}{}^l \geq -nK_1$. This is clear since ν has unit length and traces are independent of the frame. Accordingly, we have to ensure that on M_0 the mean curvature is large enough by assuming $H^2/n > nK_1$ on M_0.

Remark 4.5. Observe that this condition does not depend on the choice of the unit normal on M_0. Above, however, we have assumed that $H > 0$ on M_0. For reasons of simplicity we will keep this assumption, whence it follows that we have $H > nK_1^{1/2}$ on M_0.

By a little extra argument it now already follows that $\min_{M_0} H = H_{\min}(0) \leq H$ as long as the solution exists. For since we have demanded $H^2/n > nK_1$, there is some $\varepsilon > 0$ such that $H \geq nK_1^{1/2} + \varepsilon$ on M_0, since M_0 is compact. The maximum principle applies as long as $H > nK_1^{1/2}$. If there was some first time $t_0 > 0$ such that $H \leq nK_1^{1/2}$, we would have $H > nK_1^{1/2}$ for all time intervals $[0, t_0 - \delta]$, $\delta > 0$. Then the maximum principle applies on these intervals, yielding that $\min_{M_{(t_0-\delta)}} H \geq H_{\min}(0) \geq nK_1^{1/2} + \varepsilon$. We thus get a contradiction to the continuity of H.

We can gain even more information about H. Observe that in view of $H^2/n > nK_1$ on M_0, there is some $\delta > 0$ such that

$$H^2 \geq n^2 K_1 + n\delta^2 H^2 \quad \text{on } M_0.$$

Similar as above, the maximum principle guarantees that this relation holds as long as the solution exists. Due to $\bar{R}_{0l0}{}^l \geq -nK_1$ it thus follows from (22)

$$\frac{\partial}{\partial t} H \geq \Delta H + \delta H^3.$$

This relation can have a bounded solution only on a finite time interval, as we see now. Let φ be the solution of the ordinary differential equation

$$\frac{\partial}{\partial t}\varphi = \delta H^3, \quad \varphi(0) = H_{\min}(0) > nK_1^{1/2}.$$

If we consider φ as a function on $M \times [0, T[$, we have $\Delta\varphi = 0$ and thus

$$\frac{\partial}{\partial t}(H - \varphi) \geq \Delta(H - \varphi) + \delta(H^3 - \varphi^3).$$

A similar extra argument as above yields that the maximum principle applies, showing $H \geq \varphi$ on $[0, T[$. The function φ may now be explicitly determined. In fact it is given by

$$\varphi(t) = \frac{H_{\min}(0)}{\sqrt{1 - 2\delta H_{\min}^2(0) \cdot t}},$$

which tends to ∞ as we let $t \to \frac{1}{2}\delta^{-1}H_{\min}^{-2}(0)$. The mean curvature thus gets unbounded at this time *at the latest*.

This is a first crucial global result about the evolving surface: In the case that the sectional curvatures of N are bounded from below by $-K_1$ and if we have $H^2 > n^2K_1$ on M_0, an existing solution can only admit bounded mean curvature on a *finite time interval* $[0, T[$, $T < \infty$.

What, however, will happen to the shape of M_t as $t \to T$? For this purpose we have to study the secand fundamental form by means of a suitable maximum principle.

4.2 A maximum principle for tensors

Clearly, all information about convexity, which is determined by h_{ij}, must be buried in its evolution equation stated in lemma 3.7. As we have mentioned already, the parabolic structure of this equation suggests to apply an adaptation of the maximum principle to tensorial equations.

How could we accomplish such an adaptation? Recall that the crucial step in the proof of the ordinary parabolic maximum principle is to derive a contradiction based on the assumption that the maximum (or minimun) be attained in the interior of the domain. In particular, if we consider some smooth function f with

$$\frac{\partial}{\partial t}f > \Delta f + b^k\frac{\partial}{\partial x^k}f \quad \text{in } U :=]0, T[\times\Omega, \quad \Omega \in \mathbb{R}^n,$$

we get the contradiction by assuming the existence of a local minimum of f in U.

This idea is extended to tensorial equations in the proof of the following theorem, which is a slight variant of theorem 9.1 in [6].

In the sequel all tensor (and vector) fields are smooth.

Theorem 4.6. *Let M be a compact Riemannian manifold whose metric g_{ij} depends on the time t, for $t \in [0, T[$, $T < \infty$. Furthermore let T and S be smooth time-dependent symmetric tensor fields on M with components T_{ij} and S_{ij}, respectively, satisfying the relation*

$$\frac{\partial}{\partial t}T_{ij} \geq \Delta T_{ij} + b^k\nabla_k T_{ij} + S_{ij} \tag{23}$$

everywhere on M for all $0 \le t < T$.

Let Φ be a mapping from the set of $(2,0)$-tensors onto itself such that $S = \Phi(T)$. Suppose that Φ has the following properties: For any $\eta > 0$ and $\varepsilon > 0$ there is some $0 < C(\eta) < \infty$ such that $|\tilde{S}_{ij} - S_{ij}| \le C(\eta)\varepsilon$ with $\tilde{T}_{ij} := T_{ij} + \varepsilon g_{ij}$ and $\tilde{S} := \Phi(\tilde{T})$ on the whole interval $[0, T - \eta]$ and everywhere on M (continuity condition). Moreover, for all $0 \le \varepsilon < \varepsilon_0$, suppose that if $v = \{v^i\}$ is a null-eigenvector of $\tilde{T}$, that is, $\tilde{T}_{ij}v^iv^j = 0$ and $v \ne 0$, we then have $\tilde{S}_{ij}v^iv^j \ge 0$ (null-eigenvector condition).

Then, if $T_{ij} \ge 0$ (in the sense of definition 4.1) for $t = 0$, it remains so for all $0 \le t < T$.

Note that we have $\Delta T_{ij} = g^{kl}\nabla_k\nabla_l T_{ij}$ and the covariant derivative is to be taken in the tensorial sense.

Remark 4.7. In contrast to the strict inequality above, we only demand a weak inequality here. Moreover, we allow for an extra summand S_{ij}. The major part of the technical work in the proof consists of treating these two generalizations. The continuity condition and the null-eigenvector condition are required to make S_{ij} controllable at the crucial steps of the proof.

Proof. Let $\eta > 0$ and $\varepsilon > 0$. We define

$$\tilde{T}_{ij} := T_{ij} + \varepsilon(\delta + t)g_{ij}.$$

We assert that $\tilde{T}_{ij} > 0$ for all $0 \le t \le T - \eta$. We prove this by showing it for the interval $[0, \delta]$ for some $\delta > 0$ to be chosen. Then we repeat the argument finitely many times to eventually cover the whole interval $[0, T - \eta]$.

We have

$$\frac{\partial}{\partial t}\tilde{T}_{ij} = \frac{\partial}{\partial t}T_{ij} + \varepsilon g_{ij} + \varepsilon(\delta + t)\frac{\partial}{\partial t}g_{ij}.$$

Since M and the interval $[0, T - \eta]$ are compact and g_{ij} is smooth, $|\frac{\partial}{\partial t}g_{ij}|$ is bounded on $M \times [0, T - \eta]$ and we may choose $\delta > 0$ so small such that for all $\varepsilon > 0$

$$\frac{\partial}{\partial t}\tilde{T}_{ij} \ge \frac{\partial}{\partial t}T_{ij} + \frac{1}{2}\varepsilon g_{ij}$$

for all $t \in [0, \delta]$. Let us suppose for the moment that $S_{ij} \equiv 0$. By assumption we then have

$$\frac{\partial}{\partial t}\tilde{T}_{ij} \ge \frac{\partial}{\partial t}T_{ij} + \frac{1}{2}\varepsilon g_{ij} > \Delta T_{ij} + b^k\nabla_k T_{ij} = \Delta\tilde{T}_{ij} + b^k\nabla_k\tilde{T}_{ij} \tag{24}$$

on $[0, \delta]$.

We also have $\tilde{T}_{ij} > 0$ at $t = 0$ and we want to show that this remains so on $[0, \delta]$. For this purpose suppose that there is a first time $\theta \in [0, \delta]$ such that there is some point $p \in M$ and a vector $0 \ne v = \{v^i\} \in T_pM$ of unit length with $\tilde{T}_{ij}v^iv^j = 0$. We construct a contradiction from this situation: Locally around p we may extend v to a (smooth) vector field on $T_pM \times [0, \delta]$, which we again denote by $v = \{v^i\}$. We may choose v such that $\nabla_k v^i = 0$ at (θ, p) and such that v^i does not depend on t. By letting

$$f := \tilde{T}_{ij}v^iv^j$$

we transfer the problem to the realm of smooth functions and may proceed in analogy to the proof of the ordinary maximum principle: Since $f \geq 0$ for $0 \leq t \leq \theta$ it follows

$$\frac{\partial}{\partial t} f \leq 0 \quad \text{at} \ (\theta, p).$$

On the other hand, p is a local minimum of $f(\theta, \cdot)$, that is, we have

$$\nabla_k f = 0 \quad \text{and} \quad \Delta f \geq 0.$$

If one of these relations was not satisfied, there would be a geodesic γ through p such that $f(\theta, \gamma(q)) < 0$ for some q near p, which contradicts the minimality of θ.

As a consequence we get a contradiction to (24) by

$$\frac{\partial}{\partial t} \tilde{T}_{ij} v^i v^j = \frac{\partial}{\partial t} f \leq \Delta f + b^k \nabla_k f = \Delta \tilde{T}_{ij} v^i v^j + b^k \nabla_k \tilde{T}_{ij} v^i v^j \quad \text{at} \ (\theta, p).$$

where we have used the relations $\frac{\partial}{\partial t} f = (\frac{\partial}{\partial t} \tilde{T}_{ij}) v^i v^j$, $\nabla_k f = \nabla_k (\tilde{T}_{ij}) v^i v^j$ and

$$\Delta f = \Delta \tilde{T}_{ij} v^i v^j + 2 g^{kl} \underbrace{\tilde{T}_{ij} v^i}_{=0} \nabla_k \nabla_l v^j = \Delta \tilde{T}_{ij} v^i v^j,$$

which are due to the particular choice of v.

If we now allow for $S_{ij} \not\equiv 0$, (24) takes the form

$$\frac{\partial}{\partial t} \tilde{T}_{ij} \geq \frac{\partial}{\partial t} T_{ij} + \frac{1}{2} \varepsilon g_{ij} > \Delta T_{ij} + b^k \nabla_k T_{ij} + S_{ij} = \Delta \tilde{T}_{ij} + b^k \nabla_k \tilde{T}_{ij} + S_{ij}. \tag{25}$$

If, however, we invoke the assumptions on Φ, we may reason as follows: Since v is a null-eigenvector for $\tilde{T}_{ij}$, we have for $\tilde{S} = \Phi(\tilde{T})$ the relation $\tilde{S}_{ij} v^i v^j \geq 0$. On the other hand, observe that we have $\tilde{T}_{ij} - T_{ij} = \varepsilon(\delta + t) g_{ij}$ for $t \in [0, \delta]$. According to the assumption this implies $|S_{ij} - \tilde{S}_{ij}| \leq C(\eta)(\delta + t)\varepsilon \leq 2C(\eta)\delta\varepsilon$ for some $C(\eta) < \infty$. Altogether it follows $S_{ij} v^i v^j \geq 2C(\eta)\delta\varepsilon$. If necessary, we then decrease $\delta > 0$ that much that we have $2C(\eta)\delta\varepsilon < \varepsilon/2$. If we then plug v into (25), we get with $g_{ij} v^i v^j = 1$

$$\frac{\partial}{\partial t} \tilde{T}_{ij} v^i v^j \geq \frac{\partial}{\partial t} T_{ij} v^i v^j > \Delta T_{ij} v^i v^j + b^k \nabla_k T_{ij} v^i v^j = \Delta \tilde{T}_{ij} v^i v^j + b^k \nabla_k \tilde{T}_{ij} v^i v^j,$$

and we can construct the contradiction as above.

We thus have shown that $\tilde{T}_{ij} > 0$ on $[0, \delta]$. Repeating the argument finitely many times yields the same result for $[0, T - \eta]$. We now let $\varepsilon \to 0$ and thus obtain $T_{ij} \geq 0$ on $[0, T - \eta]$. The theorem follows by letting $\eta \to 0$. □

4.3 Preservation of modified convexity

Theorem 4.6 is a crucial tool we want apply to the evolution equation of the second fundamental form given by

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} \quad &= \quad \Delta h_{ij} - 2H h_j^m h_{im} + |A|^2 h_{ij} + h_{ij} \bar{R}_{0l0}{}^l \\ &\quad - h_{jl} \bar{R}^l{}_{mi}{}^m - h_{il} \bar{R}^l{}_{mj}{}^m + 2 h_{lm} \bar{R}^l{}_i{}^m{}_j - \bar{\nabla}_j \bar{R}_{0li}{}^l - \bar{\nabla}_l \bar{R}_{0ij}{}^l. \end{aligned} \tag{26}$$

We therefore have to control all quantities involved such that the conditions of the theorem are satisfied.

Example 4.8. For motivational reasons suppose for the moment that the ambient space be Euclidean. Then both $\bar{R}$ and $\bar{\nabla}\bar{R}$ vanish and (26) takes the form

$$\frac{\partial}{\partial t}h_{ij} \;=\; \Delta h_{ij} - 2Hh_j^m h_{im} + |A|^2 h_{ij}. \tag{27}$$

Setting $T_{ij} := h_{ij}$, $b^k \equiv 0$ and $S_{ij} := -2Hh_{jl}h_{im}g^{lm} + |A|^2 h_{ij}$ shows that according to theorem 4.6 an initially convex hypersurface (i.e. $h_{ij} \geq 0$ at $t = 0$) remains convex on the interval $I = [0, T[$ (i.e. $h_{ij} \geq 0$ for $0 \leq t < T$). This is clear since S_{ij} is a polynomial in h_{ij} formed by contracting products of h_{ij} by the metric g_{ij}. Thus it satisfies the continuity condition in theorem 4.6. Moreover, the null-eigenvector condition becomes trivial due to $h_{ij}v^i v^j > 0$ for all $v \neq 0$ - there are no null-eigenvectors of T_{ij}. We thus see that we indeed can use the maximum principle to show that convexity is preserved, at least in the special case of Euclidean ambient space.

Now we turn our interest to general Riemannian ambient spaces. It will turn out that in contrast to the above example we do not succeed by simply setting $T_{ij} := h_{ij}$. Instead we first provide a basic technical preparation which enables us to apply theorem 4.6 in several different cases. Note that in the following considerations all tensors are defined on $M \times [0, T[$.

We define the symmetric tensor

$$T_{ij} := \frac{h_{ij}}{H} - fg_{ij}$$

for some smooth function f to be chosen. Our goal is to write down the parabolic differential equation (23) for some b^k and S_{ij}.

We need two technical lemmas.

Lemma 4.9. *For a smooth function φ and a smooth tensor field M_{ij} we have*

$$\Delta \frac{M_{ij}}{\varphi} = \frac{\varphi \Delta M_{ij} - M_{ij}\Delta\varphi}{\varphi^2} - \frac{2}{\varphi} g\left(\nabla\varphi, \nabla\left(\frac{M_{ij}}{\varphi}\right)\right).$$

Proof. Direct calculation. □

Remark 4.10. We have used the notation $g\left(\nabla\varphi, \nabla\left(\frac{M_{ij}}{\varphi}\right)\right) := g^{kl}\nabla_k\varphi\nabla_l\left(\frac{M_{ij}}{\varphi}\right)$.

Lemma 4.11. *We have*

$$\begin{aligned} \frac{\partial}{\partial t}\frac{1}{H^\alpha} \;&=\; \Delta\left(\frac{1}{H^\alpha}\right) + \frac{2}{H} g\left(\nabla H, \nabla\left(\frac{1}{H^\alpha}\right)\right) \\ &\quad -\alpha(\alpha-1)\frac{1}{H^{\alpha+2}}|\nabla H|^2 - \frac{\alpha}{H^\alpha}(|A|^2 + \bar{R}_{0l0}{}^l). \end{aligned}$$

Proof. We have

$$\frac{\partial}{\partial t}\frac{1}{H^\alpha} = \frac{\alpha}{H^{\alpha+1}}\frac{\partial}{\partial t}H = \frac{\alpha}{H^{\alpha+1}}(\Delta H + H(|A|^2 + \bar{R}_{0l0}{}^l))$$

and

$$\Delta\left(\frac{1}{H^\alpha}\right) = g^{kl}\nabla_k\left(\frac{-\alpha}{H^{\alpha+1}}\nabla_l H\right) = \frac{-\alpha}{H^{\alpha+1}}\Delta H + \alpha(\alpha+1)\frac{1}{H^{\alpha+2}}|\nabla H|^2.$$

Thus we get

$$\frac{\partial}{\partial t}\frac{1}{H^\alpha} = \Delta\left(\frac{1}{H^\alpha}\right) - \alpha(\alpha+1)\frac{1}{H^{\alpha+2}}|\nabla H|^2 - \frac{\alpha}{H^\alpha}(|A|^2 + \bar{R}_{0l0}{}^l).$$

From this and

$$\frac{2}{H}\left\langle \nabla H, \nabla(\frac{1}{H^\alpha})\right\rangle = -2\frac{\alpha}{H^{\alpha+2}}|\nabla H|^2$$

the assertion follows. □

Returning to T_{ij} defined above we get by lemma 4.9

$$\Delta T_{ij} = \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} - \frac{2}{H}g\left(\nabla H, \nabla\left(\frac{h_{ij}}{H}\right)\right) - g_{ij}\Delta f. \tag{28}$$

Moreover, by lemma 4.11 and by the evolution equation of the second fundamental form (26) we compute using $\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij}$

$$\begin{aligned}\frac{\partial}{\partial t}T_{ij} &= \frac{\Delta h_{ij}}{H} + h_{ij}\Delta\frac{1}{H} + \frac{U_{ij}}{H} \\ &\quad + h_{ij}\frac{2}{H}g\left(\nabla H, \nabla\frac{1}{H}\right) - \frac{h_{ij}}{H}(|A|^2 + \bar{R}_{0l0}{}^l) - \frac{\partial}{\partial t}fg_{ij} + 2fHh_{ij},\end{aligned} \tag{29}$$

where for reasons of clearness we comprise the terms of the right hand side of (26) without the Laplacian by $U_{ij} := \frac{\partial}{\partial t}h_{ij} - \Delta h_{ij}$.

One easily verfies that we have

$$\Delta\frac{1}{H} = -\frac{\Delta H}{H^2} - \frac{2}{H}g\left(\nabla H, \nabla\frac{1}{H}\right).$$

By this identity we derive from (29)

$$\begin{aligned}\frac{\partial}{\partial t}T_{ij} &= \frac{H\Delta h_{ij} - h_{ij}\Delta H}{H^2} + \frac{1}{H}U_{ij} - \frac{h_{ij}}{H}(|A|^2 + \bar{R}_{0l0}{}^l) \\ &\quad - \frac{\partial}{\partial t}fg_{ij} + 2fHh_{ij}.\end{aligned} \tag{30}$$

By finally putting (28) and (30) together we obtain an equation similar to the one required for the tensorial maximum principle:

$$\begin{aligned}\frac{\partial}{\partial t}T_{ij} &= \Delta T_{ij} + \frac{2}{H}g(\nabla H, \nabla T_{ij}) + \frac{2}{H}g(\nabla H, \nabla(f)g_{ij}) + g_{ij}\Delta f \\ &\quad + \frac{1}{H}U_{ij} - \frac{h_{ij}}{H}(|A|^2 + \bar{R}_{0l0}{}^l) - \frac{\partial}{\partial t}fg_{ij} + 2fHh_{ij}.\end{aligned} \tag{31}$$

Indeed the structure of this equation matches with the one treated in (23), where $b^k = \frac{2}{H}\nabla_l H g^{lk}$ and

$$\begin{aligned}S_{ij} &= \frac{2}{H}g(\nabla H, \nabla(f)g_{ij}) + g_{ij}\Delta f + \frac{1}{H}U_{ij} \\ &\quad - \frac{h_{ij}}{H}(|A|^2 + \bar{R}_{0l0}{}^l) - \frac{\partial}{\partial t}fg_{ij} + 2fHh_{ij}.\end{aligned} \tag{32}$$

This preparation was useful since we now can study the nature of S_{ij} dependent on the choice of f. In particular we have to examine it in view of the conditions demanded in theorem 4.6.

For the moment let us ignore the null-eigenvector condition and turn our interest to the continuity condition. We have to consider $\tilde{T}_{ij} = T_{ij} + \varepsilon g_{ij}$ and show that for any small $\eta > 0$ there is a constant $C(\eta)$ such that $|\tilde{S}_{ij} - S_{ij}| < C(\eta)\varepsilon$. Observe that we get $\tilde{S}_{ij}$ by replacing f by $f + \varepsilon$ in (32). We thus get $|\tilde{S}_{ij} - S_{ij}| = 2\varepsilon H|h_{ij}|$. Since H and $|h_{ij}|$ are bounded on $[0, T - \eta]$, the continuity condition is satisfied.

It now only remains to appropriately choose the function f such that the second condition of the tensorial maximum principle 4.6 - the null-eigenvector condition for S_{ij} - is satisfied. Then the maximum principle applies, stating that the relation $T_{ij} \geq 0$ is valid as long as the mean curvature flow goes on, provided that it holds on M_0.

Applying this statement to our definition of T_{ij}, we are able to show that for appropriate choices of f relations of the form

$$Hh_{ij} \geq H^2 f g_{ij} \tag{33}$$

remain true as long as the flow goes on, whenever they are satisfied at $t = 0$. (Of course the choice of f has to imply the null-eigenvector condition for S_{ij}.)

Remark 4.12. Observe that equation (33) is equivalent to $h_{ij} \geq Hfg_{ij}$, provided that $H > 0$. For $f \equiv 0$ this is just the definition of convexity. For other choices of f, however, the original notion of convexity gets modified. For example, earlier we have assumed $H^2 > n^2 K_1$, $K_1 > 0$, to show that the solution exists only on a finite time interval. This implies $f \geq nK_1/H^2$, including an ambient curvature term in the convexity condition. The result gets less powerful the more terms are involved. However, the maximum principle demands that we have to choose f in such a way that the null-eigenvector condition is satisfied.

Since U_{ij} and thus S_{ij} contain terms involving the ambient curvature tensor and its gradient, we have to somehow control both quantities by f in order to guarantee the null-eigenvector condition for S_{ij}, as we will see. In addition to the lower bound $-K_1$ of the sectional curvatures we therefore impose another obstruction on the geometry of N.

We demand that $|\bar{\nabla}\bar{R}|^2 \leq L^2$ *for some* $L \geq 0$.

Then it turns out that we succeed if we demand that the relation

$$Hh_{ij} > nK_1 g_{ij} + \frac{n^2}{H} L g_{ij} \tag{34}$$

be satisfied on M_0. This is our modified convexity condition, which we want to show to be preserved during the flow.

Remark 4.13. Contracting this relation yields $H^2 > n^2 K_1 + n^3 L/H$. For a suitable choice of unit normal this implies $H^2 > n^2 K_1$, our earlier assumption, which we have proved to be preserved. Moreover, for this choice of normal H is bounded from below by $\min_{M_0} H = H_{\min}(0) > n(K_1)^{1/2}$, since M_0 is compact.

Observe that due to the strict inequality in (34) there is some $0 < \varepsilon < 1/n$ such that

$$Hh_{ij} \geq nK_1 g_{ij} + \frac{n^2}{H} L g_{ij} + \varepsilon(H^2 - n^2 K_1) g_{ij} \quad \text{on } M_0. \tag{35}$$

Accordingly we choose

$$f := \varepsilon + \frac{1-n\varepsilon}{H^2} nK_1 + \frac{n^2}{H^3} L.$$

For this choice of f we now prove that the null-eigenvector condition of (4.6) is satisfied for S_{ij}.

To this end observe that we have to vary the tensor T_{ij} by defining $\tilde{T}_{ij} = T_{ij} + \varepsilon g_{ij}$ for strictly positive $0 \le \varepsilon < \varepsilon_0$ and show that the tensor $\tilde{S} = \Phi(\tilde{T})$ possesses the null-eigenvector property. It is therefore sufficient to verify the null-eigenvector property for all $T_{ij} = h_{ij}/H - f g_{ij}$ with choices of ε for which (35) still is valid, and we accordingly choose ε_0 small enough.

For any $0 \le \varepsilon < \varepsilon_0$ we then consider the first time t_0 where at some point $p \in M_{t_0}$ a zero eigenvector v of T_{ij} occurs. We have to show that $S_{ij} v^i v^j \ge 0$ at p. Since this is a purely pointwise relation, we may choose a convenient basis for $T_p M_{t_0}$. Therefore we choose a basis $\{e_i\}$ such that $g_{ij} = \delta_{ij}$ and h_{ij} becomes diagonal. Obviously T_{ij} then is diagonal as well, according to its definition. Moreover we may suppose that the Christoffel symbols all vanish at p. We further assume that $v = e_1$. Finally let $\kappa_1, \dots, \kappa_n$ be the eigenvalues of h_{ij} at p. Because of $v = e_1$ it is $T_{11} = 0$ and it follows

$$\kappa_1 = Hf = H\varepsilon + \frac{1-n\varepsilon}{H} nK_1 + \frac{n^2}{H^2} L.$$

If we compute $\frac{\partial}{\partial t} f$, ∇f and Δf and plug these terms into (32), we obtain

$$\begin{aligned} S_{ij} \;=\; & -2h_{il}h^l_j + 2\varepsilon H h_{ij} + \frac{2n(1-n\varepsilon)}{H} K_1 h_{ij} + \frac{2n^2}{H^2} L h_{ij} + \frac{2n(1-n\varepsilon)}{H^4} K_1 |\nabla H|^2 g_{ij} \\ & + \frac{6n^2}{H^5} L |\nabla H|^2 g_{ij} + \frac{1}{H}(2h_{lm}\bar{R}^l{}_i{}^m{}_j - h_{jl}\bar{R}^l{}_{mi}{}^m - h_{il}\bar{R}^l{}_{mj}{}^m) - \frac{1}{H}(\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l) \\ & + \left(\frac{2n(1-n\varepsilon)}{H^2} K_1 + \frac{3n^2}{H^3} L \right) (|A|^2 + \bar{R}_{0l0}{}^l) g_{ij}. \end{aligned}$$

We then consider $S_{ij} v^i v^j = S_{11}$ and employ that in our frame we have the relations

$$-\frac{1}{H}(\bar{\nabla}_1 \bar{R}_{0l1}{}^l + \bar{\nabla}_l \bar{R}_{011}{}^l) \ge -\frac{2n}{H} L.$$

and

$$-2h_{1l}h^l_1 + 2\varepsilon H h_{11} + \frac{2n(1-n\varepsilon)}{H} K_1 h_{11} + \frac{2n^2}{H^2} L h_{11} = -2HM_{11}h_{11} = 0$$

and

$$\frac{2n(1-n\varepsilon)}{H^4} K_1 |\nabla H|^2 g_{11} + \frac{6n^2}{H^5} L |\nabla H|^2 g_{11} \ge 0$$

and

$$\frac{1}{H}(2h_{lm}\bar{R}^l{}_1{}^m{}_1 - h_{1l}\bar{R}^l{}_{m1}{}^m - h_{1l}\bar{R}^l{}_{m1}{}^m) = \frac{2}{H} \sum_{l=2}^n \bar{R}_{1l1l}(\kappa_l - \kappa_1)$$

as well as $|A|^2 \ge H^2/n$ and $\bar{R}_{0l0}{}^l \ge -nK_1$.

Combining all these relation and applying them to S_{11} yields

$$S_{11} \ge \frac{2}{H} \sum_{l=2}^n \bar{R}_{1l1l}(\kappa_l - \kappa_1) - \frac{2n}{H} L + 2(1-n\varepsilon)K_1 + \frac{3n}{H} L - \frac{2n^2(1-n\varepsilon)}{H^2} K_1^2 - \frac{3n^3}{H^3} LK_1. \quad (36)$$

Now observe that by our agreement that all ambient sectional curvatures be bounded from below by $-K_1$, we are allowed to further estimate

$$\frac{2}{H}\sum_{l=2}^{n}\bar{R}_{1l1l}(\kappa_l-\kappa_1) \geq -\frac{2}{H}K_1\sum_{l=2}^{n}(\kappa_l-\kappa_1) = -\frac{2}{H}K_1(H-n\kappa_1).$$

If we now plug in the value for κ_1 and use this estimate with (36), we finally find

$$S_{ij}v^iv^j = S_{11} \geq \frac{n}{H}L(1-\frac{n^2}{H^2}K_1),$$

which is non-negative due to the fact that we have found $H^2 \geq n^2K_1$ as long as the solution exists (remark 4.13).

Thus S_{ij} satisfies the null-eigenvector condition; the maximum principle applies and we conclude that the relation

$$Hh_{ij} \geq nK_1g_{ij} + \frac{n^2}{H}Lg_{ij} + \varepsilon(H^2-n^2K_1)g_{ij}$$

remains true for the same $0 < \varepsilon < 1/n$ as long as the solution exists, that is on $[0,T[$.

We summarize the results of this chapter.

Theorem 4.14. *Let M_0 be a closed hypersurface which is smoothly immersed in a Riemannian manifold N. Suppose that all sectional curvatures of N are bounded from below by $-K_1$ for some $K_1 \geq 0$. If we have $H > n(K_1)^{1/2}$ on M_0, it follows that the mean curvature flow of M_0 can have a solution only on a finite time interval $[0,T[$.*

Furthermore suppose that $|\bar{\nabla}\bar{R}|^2 \leq L^2$ for some $L \geq 0$. If on M_0 we have the relation

$$Hh_{ij} > nK_1g_{ij} + \frac{n^2}{H}Lg_{ij},$$

then this remains true on the whole interval $[0,T[$.

Remark 4.15. The above proof has shown that for our particular choice of f the null-eigenvector condition of S_{ij} is satisfied. However, we had to include terms involving the bounds K_1 and L of ambient curvature. One might doubt if this is really necessary, or if they might have been avoided by using sharper arguments. In particular the question arises, if proper convexity (and not the modified convexity from above) is preserved, too, even in the case when K_1 or L do not vanish. We will discuss this question in the chapters 6 and 7.

Before that, however, we examine how the relation of the principal curvatures changes during the flow, particularly when t approaches the singularity $T < \infty$ of H.

5 The pinching of the principal curvatures

In the last chapter we have learned that for the maximal mean curvature flow solution of smooth compact initial surfaces M_0 without boundary the relation

$$Hh_{ij} > nK_1 g_{ij} + \frac{n^2}{H} L g_{ij}$$

is preserved, if true on M_0. Moreover, we have seen that the solution only exists on the finite time interval $[0, T[$, $T < \infty$.

In the present chapter we examine how in this case the principal curvatures κ_i, $i = 1 \ldots, n$ of M_t are related to each other, especially for $t \to T$. The reason why we restrict our discussion to this case is that we then are able to invoke the results of the previous chapter. As it will turn out, the proceeding mean curvature flow forces the principal curvatures to get pinched in a sense which will be specified below. This is another major result of [8], where, however, the technical details of the proof are largely omitted. We have therefore decided to illuminate the employed techniques in detail in order to give a complete picture of this extensive proof.

Remark 5.1. The proof uses a whole string of different estimates, many of which take advantage of the assumptions and results of the previous chapters. As said above, *we therefore keep all assumptions both on M_0 and N of theorem 4.14 throughout the entire chapter.* In particular we then have the bounds K_1 and L of the ambient curvature terms at our disposal, as well as the fact that for some $0 < \varepsilon < 1/n$ relation (35) is satisfied during the flow, as we have seen in the proof of theorem 4.14. Finally recall that we have chosen the unit normal such that we have $H > 0$ on $M \times [0, T[$. All these assumptions will be employed in the following arguments.

5.1 The problem

Let $\kappa_1, \ldots, \kappa_n$ be the eigenvalues of the second fundamental form h_{ij} of M_t. Evidently the quantity

$$\frac{1}{n}\sum_{i<j}^{n}(\kappa_i - \kappa_j)^2 \tag{37}$$

measures how far the κ_i diverge from each other.

Lemma 5.2. *We have*

$$\frac{1}{n}\sum_{i<j}^{n}(\kappa_i - \kappa_j)^2 = |A|^2 - \frac{1}{n}H^2.$$

Proof. At $p \in M_t$ we choose coordinates such that $g_{ij} = \delta_{ij}$ and $h_{ij} = \mathrm{diag}(\kappa_1, \ldots, \kappa_n)$ becomes diagonal. Then

$$\begin{aligned} n|A|^2 - H^2 &= n\sum_{i=1}^{n}\kappa_i^2 - (\sum_{i=1}^{n}\kappa_i)^2 = n\sum_{i=1}^{n}\kappa_i^2 - \sum_{i=1}^{n}\kappa_i^2 - \sum_{i<j}2\kappa_i\kappa_j \\ &= (n-1)\sum_{i=1}^{n}\kappa_i^2 - \sum_{i<j}2\kappa_i\kappa_j = \sum_{i<j}(\kappa_i^2 - 2\kappa_i\kappa_j + \kappa_j^2) = \sum_{i<j}(\kappa_i - \kappa_j)^2. \end{aligned}$$

□

Remark 5.3. It is always $|A|^2 - H^2/n \leq H^2$. For, in the setting of the proof above, this is equivalent to $n\sum\kappa_i^2 \leq (n+1)(\sum\kappa_i)^2$, which is true if all $\kappa_i \geq 0$. This, in turn, is a fact which directly follows from relation (35).

According to the remark it is clear that the function

$$f_\sigma := \frac{|A|^2 - \frac{1}{n}H^2}{H^{2-\sigma}}, \quad f_\sigma : M_t \to \mathbb{R}, \tag{38}$$

for $\sigma = 0$ is bounded uniformly in t. It is, in turn, *not* trivial to answer this question for f_σ, $\sigma > 0$. In other words: The principal curvatures indeed approach each other, if there is an (arbitrarily small) $\sigma > 0$ such that f_σ is bounded uniformly in t.

Our goal in this chapter is therefore to prove the existence of some $\sigma > 0$ such that f_σ is bounded uniformly in t.

Remark 5.4. In view of the extent of the proof it is advisable to briefly outline the structure of the proof. At the beginning we derive an evolution equation for f_σ similar to the ones developped earlier for other quantities. This equation is improved by several estimates taking advantage of the assumptions listed in remark 5.1. The result is employed to conclude that sufficiently high p-norms of f_σ are bounded uniformly in time. An argument using a general Sobolev inequality along with an iteration method finally yields the desired bound. As we will see, the arguments demand two additional assumptions on the geometry of N: an upper bound for the sectional curvature as well as a strictly positive lower bound for the injectivity radius.

5.2 The evolution equation for f_σ

Let $\sigma > 0$ from now on. As we want to study the behaviour of f_σ during the flow, we have to look at its evolution equation.

In view of the evolution equations derived earlier, we expect it to be of the form

$$\frac{\partial}{\partial t}f_\sigma = \Delta f_\sigma + b^k\nabla_k f_\sigma + cf_\sigma + d. \tag{39}$$

We therefore compute the quantities $\frac{\partial}{\partial t}f_\sigma$, Δf_σ, and $\nabla_k f_\sigma$ in order to determine b^k, c, and d.

To compute the time derivative $\frac{\partial}{\partial t}f_\sigma$ we take advantage of the evolution equations of $|A|^2$ and H given in lemma 3.11 and lemma 3.10, respectively. Let $\alpha := 2-\sigma$. Based on the definition of f_σ we compute

$$\begin{aligned}
\frac{\partial}{\partial t}f_\sigma &= \frac{1}{H^\alpha}[\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + \bar{R}_{0l0}{}^l) \\
&\quad -4(h^{ij}h_j^m\bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj}) \\
&\quad -2h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^l + \bar{\nabla}_l\bar{R}_{0ij}{}^l)] - \alpha\frac{|A|^2}{H^{\alpha+1}}(\Delta H + H(|A|^2 + \bar{R}_{0l0}{}^l)) \\
&\quad -(2-\alpha)\frac{H^{1-\alpha}}{n}(\Delta H + H(|A|^2 + \bar{R}_{0l0}{}^l)) \\
&= \frac{H\Delta|A|^2 - \alpha|A|^2\Delta H}{H^{\alpha+1}} - \frac{2-\alpha}{n}H^{1-\alpha}\Delta H - \frac{2}{H^\alpha}|\nabla A|^2 \\
&\quad +(|A|^2 + \bar{R}_{0l0}{}^l)(2-\alpha)f_\sigma - \frac{4}{H^\alpha}(h^{ij}h_j^m\bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj}) \\
&\quad -\frac{2}{H^\alpha}h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^l + \bar{\nabla}_l\bar{R}_{0ij}{}^l).
\end{aligned} \tag{40}$$

Note that in the second identity we have used the definition of f_σ to collect terms. On the other hand we have

$$\nabla_k f_\sigma = \frac{\nabla_k|A|^2}{H^\alpha} - \alpha\frac{|A|^2}{H^{\alpha+1}}\nabla_k H - (2-\alpha)\frac{H^{1-\alpha}}{n}\nabla_k H. \tag{41}$$

If we covariantly differentiate (41) once more and contract, we get

$$\begin{aligned}\Delta f_\sigma &= \frac{H\Delta|A|^2 - \alpha|A|^2\Delta H}{H^{\alpha+1}} - \frac{2-\alpha}{n}H^{1-\alpha}\Delta H - 2\frac{\alpha}{H^{\alpha+1}}\left\langle \nabla|A|^2, \nabla H\right\rangle \\ &\quad +|\nabla H|^2\left[\alpha(\alpha+1)\frac{|A|^2}{H^{\alpha+2}} - (2-\alpha)(1-\alpha)\frac{H^{-\alpha}}{n}\right].\end{aligned} \tag{42}$$

Furthermore from (41) it follows

$$\langle \nabla f_\sigma, \nabla H\rangle = \frac{\langle \nabla|A|^2, \nabla H\rangle}{H^\alpha} - \frac{\alpha|A|^2}{H^{\alpha+1}}|\nabla H|^2 - (2-\alpha)\frac{H^{1-\alpha}}{n}|\nabla H|^2. \tag{43}$$

Now we collect terms and write down the evolution equation of f_σ in the form of (39). It turns out, however, that in view of the subsequent estimates it is handy to further simplify the equation by the following lemma to the final form

$$\begin{aligned}\frac{\partial}{\partial t}f_\sigma &= \Delta f_\sigma + \frac{2(\alpha-1)}{H}\left\langle \nabla H, \nabla f_\sigma\right\rangle \underbrace{+(2-\alpha)(|A|^2+\bar{R}_{0l0}{}^{l})f_\sigma}_{[1]} \\ &\quad \underbrace{-\frac{2}{H^{\alpha+2}}|H\nabla_i h_{kl} - h_{kl}\nabla_i H|^2}_{[2]} \underbrace{-\frac{(2-\alpha)(\alpha-1)}{H^{\alpha+2}}\left(|A|^2 - \frac{1}{n}H^2\right)|\nabla H|^2}_{[3]} \\ &\quad \underbrace{-\frac{4}{H^\alpha}(h^{ij}h_{jl}\bar{R}^{l}{}_{mi}{}^{m} - h^{ij}h^{lm}\bar{R}_{iljm})}_{[4]} \underbrace{-\frac{2}{H^\alpha}h^{ij}(\bar{\nabla}_j\bar{R}_{0li}{}^{l} + \bar{\nabla}_l\bar{R}_{0ij}{}^{l})}_{[5]}.\end{aligned} \tag{44}$$

Lemma 5.5. *It is*

$$|H\nabla_i h_{kl} - h_{kl}\nabla_i H|^2 = H^2|\nabla A|^2 + |A|^2|\nabla H|^2 - \left\langle \nabla|A|^2, \nabla H\right\rangle H.$$

Proof. The assertion follows from

$$|H\nabla_i h_{kl} - h_{kl}\nabla_i H|^2 = H^2|\nabla_i h_{kl}|^2 + |A|^2|\nabla H|^2 - 2Hg^{im}g^{kn}g^{ls}h_{kl}\nabla_i H\nabla_m h_{ns}$$

and

$$\begin{aligned}2g^{im}g^{kn}g^{ls}h_{kl}\nabla_i H\nabla_m h_{ns} &= g^{im}g^{kn}g^{ls}(h_{kl}\nabla_m h_{ns} + h_{ns}\nabla_m h_{kl})\nabla_i H \\ &= g^{im}\nabla_m(g^{kn}g^{ls}h_{kl}h_{ns})\nabla_i H = \left\langle \nabla|A|^2, \nabla H\right\rangle.\end{aligned}$$

□

Remark 5.6. In view of the parabolic structure of (44) the question arises if it is possible to directly derive a bound for f_σ by means of the maximum principle, similarly to how we have bounded H from below in the last chapter. This is, however, not possible, since the maximum principle requires that the coefficient of f_σ be negative or zero. In our case this is $(2-\alpha)(|A|^2 + \bar{R}_{0l0}{}^{l})$ which is far away from being negative. We thus have to use other methods.

Observe that the evolution equation contains terms involving h_{ij} as well as terms involving the ambient curvature and its gradient. We exploit this in the following section as a possibility to improve evolution equation (44) by our earlier assumptions and results.

5.3 Improving the evolution equation

Our aim in this section is to improve the evolution equation of f_σ. In particular, we derive an differential inequality for f_σ by employing the assumptions made in remark 5.1. Basically, these are the validity of estimate (35), which is a means to estimate terms involving h_{ij} uniformly in $t \in [0, T[$, and the existence of the lower bounds $-K_1$ of the ambient sectional curvatures and the bound L of the curvature gradient.

In addition to that, we from now on assume that the ambient sectional curvature be bounded from above by $K_2 > 0$.

We intend to estimate the terms [1] to [5] of the evolution equation (44) from above. For this purpose first observe that term [3] is always negative due to lemma 5.2 and thus can be omitted.

For term [2] we exploit the fact that it contains h_{ij}. Consequently we may apply relation (35), aiming to get an estimate involving the gradient of H as well as ε. This is indeed the case, as the following lemma shows.

Lemma 5.7. *The assumptions listed in remark 5.1 imply that the relation*

$$|H\nabla_i h_{kl} - h_{kl}\nabla_i H|^2 \geq \frac{1}{4}\varepsilon^2 H^2 |\nabla H|^2 - \frac{c_n}{\varepsilon}\max\{K_1^2, K_2^2\} H^2 \tag{45}$$

holds everywhere on M_t for all $t \in [0, T[$. c_n is a constant depending only on n.

Proof. We have

$$\begin{aligned} |H\nabla_i h_{kl} - h_{kl}\nabla_i H|^2 &= |\frac{1}{2}H(\nabla_i h_{kl} + \nabla_k h_{il}) - \frac{1}{2}(h_{kl}\nabla_i H + h_{il}\nabla_k H) \\ &\quad + \frac{1}{2}H(\nabla_i h_{kl} - \nabla_k h_{il}) - \frac{1}{2}(h_{kl}\nabla_i H - h_{il}\nabla_k H)|^2. \end{aligned}$$

The first and the second summand are symmetric in i and k, whereas the third and the fourth summand are antisymmetric in i and k, whence the above is equal to

$$\begin{aligned} & \frac{1}{4}|(H\nabla_i h_{kl} + \nabla_k h_{il}) - (h_{kl}\nabla_i H + h_{il}\nabla_k H)|^2 \\ + & \frac{1}{4}|(H\nabla_i h_{kl} - \nabla_k h_{il}) - (h_{kl}\nabla_i H - h_{il}\nabla_k H)|^2 \\ \geq & \frac{1}{4}|H(\nabla_i h_{kl} - \nabla_k h_{il}) - (h_{kl}\nabla_i H - h_{il}\nabla_k H)|^2 \\ = & \frac{1}{4}|H\bar{R}_{0lki} - (h_{kl}\nabla_i H - h_{il}\nabla_k H)|^2, \end{aligned} \tag{46}$$

where in the last identity we have applied the Codazzi equations $\nabla_i h_{kl} - \nabla_k h_{il} = \bar{R}_{0lki}$.

Since the desired estimate is a pointwise one and since $|\cdot|^2$ does not depend on the choice of basis, we now may choose a suitable reference frame in order to simplify further estimations. Choose for instance an orthonormal frame $e_1, \ldots, e_n$ at $p \in M_t$ such that $e_1 = \nabla H / |\nabla H|$ for the case that $|\nabla H| > 0$. (The case $|\nabla H| = 0$ is trivial and not considered here.) Then all components of ∇H vanish except for $\nabla_1 H = |\nabla H|$.

Let U be a linear subspace of $T_p M_t$ and let $U^\perp$ be its orthogonal complement. It is an obvious fact that any tensor T as a multilinear map on $T_p M_t$ may be decomposed into the direct sum of its orthogonal projections S and $S^\perp$ onto U and $U^\perp$, respectively. We then have $g(S, S^\perp) = 0$. It follows $|T|^2 = |S|^2 + |S^\perp|^2$ and thus $|T|^2 \geq |S|^2$. In our case we choose $U := \text{span}(e_1, e_2)$ and consequently find that (46) is equal to or larger than

$$\begin{aligned} & \frac{1}{4}(\bar{R}_{0112}H + |\nabla H|h_{21})^2 + \frac{1}{4}(\bar{R}_{0121}H - |\nabla H|h_{21})^2 \\ + \ & \frac{1}{4}(\bar{R}_{0212}H + |\nabla H|h_{22})^2 + \frac{1}{4}(\bar{R}_{0221}H - |\nabla H|h_{22})^2 \\ \geq \ & \frac{1}{4}(\bar{R}_{0212}H + |\nabla H|h_{22})^2 + \frac{1}{4}(\bar{R}_{0221}H - |\nabla H|h_{22})^2 \\ = \ & \frac{1}{2}(\bar{R}_{0212}H + |\nabla H|h_{22})^2. \end{aligned}$$

Now we have reached the point where we may apply relation (35), which in our frame takes the form

$$Hh_{22} \geq nK_1 + \frac{n^2}{H}L + \varepsilon(H^2 - n^2K_1) \geq \varepsilon H^2 + nK_1(1 - n\varepsilon) \geq \varepsilon H^2,$$

since $0 < \varepsilon < 1/n$. This implies that the above is greater than or equal to

$$\frac{1}{2}\bar{R}^2_{0212}H^2 + \frac{1}{2}|\nabla H|^2\varepsilon^2 H^2 + |\nabla H|h_{22}H\bar{R}_{0212}.$$

If $\bar{R}_{0212} \geq 0$ the assertion is immediate. Otherwise, because of $0 \leq h_{22} < H$, we have

$$|\nabla H|h_{22}H\bar{R}_{0212} \geq |\nabla H|H^2\bar{R}_{0212} \geq -\frac{1}{4}\varepsilon^2 H^2|\nabla H|^2 - \frac{1}{\varepsilon^2}H^2\bar{R}^2_{0212},$$

according to Cauchy's inequality with η given by

$$ab \leq \frac{1}{2\eta}a^2 + \frac{\eta}{2}b^2 \quad \text{for } a, b \in \mathbb{R} \text{ and } \eta > 0.$$

Then we further estimate

$$|H\nabla_i h_{kl} - h_{kl}\nabla_i H|^2 \geq \frac{1}{4}\varepsilon^2 H^2|\nabla H|^2 - \frac{1}{\varepsilon^2}H^2\bar{R}^2_{0212} \geq \frac{1}{4}\varepsilon^2 H^2|\nabla H|^2 - \frac{c_n}{\varepsilon^2}\max\{K_1^2, K_2^2\}H^2,$$

and the assertion follows. □

Remark 5.8. Observe that here we have used that the ambient sectional curvatures be bounded from both below and above by $-K_1$ and K_2, respectively. The second estimate is justified by the possibility of recovering $\bar{R}_{0212}$ by means of sectional curvatures (see for example the proof of lemma 3.3 of [2], p. 95)

Now we focus on terms [4] and [5] of (44), which are terms contributing information about the ambient curvature and its gradient. Thus we may use the bounds K_1, K_2 and L to derive upper estimates.

To handle term [4] we once more introduce suitable coordinates: At a fixed point we assume that $h_{ij} = \kappa_i \delta_{ij}$, where κ_i are the eigenvalues of h_{ij}. In this setting we have

$$
\begin{aligned}
& h^{ij} h_{jl} \bar{R}^l{}_{mi}{}^m - h^{ij} h^{lm} \bar{R}_{iljm} = \sum_{l,m} (\kappa_l^2 \bar{R}_{lmlm} - \kappa_l \kappa_m \bar{R}_{lmlm}) \\
= \ & \frac{1}{2} \left[\sum_{l,m} (\kappa_l^2 - \kappa_l \kappa_m) \bar{R}_{lmlm} + \sum_{l,m} (\kappa_m^2 - \kappa_l \kappa_m) \bar{R}_{lmlm} \right] \\
= \ & \frac{1}{2} \sum_{l,m} (\kappa_l - \kappa_m)^2 \bar{R}_{lmlm} = \sum_{l<k} (\kappa_l - \kappa_m)^2 \bar{R}_{lmlm} \\
\geq \ & -K_1 \sum_{l<k} (\kappa_l - \kappa_m)^2 = -nK_1 \left(|A|^2 - \frac{1}{n} H^2 \right).
\end{aligned}
$$

By the definition of f_σ we thus conclude

$$
-\frac{4}{H^\alpha} (h^{ij} h_{jl} \bar{R}^l{}_{mi}{}^m - h^{ij} h^{lm} \bar{R}_{iljm}) \leq nK_1 f_\sigma. \tag{47}
$$

Now consider term [5]. We define the *traceless second fundamental form* by

$$
\tilde{h}_{ij} := h_{ij} - \frac{H}{n} g_{ij}.
$$

The trace of term [5] with respect to i, j vanishes due to

$$
\begin{aligned}
& g^{ij} (\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l) = g^{ij} g^{lm} (\bar{\nabla}_j \bar{R}_{0lim} + \bar{\nabla}_l \bar{R}_{0ijm}) \\
= \ & g^{ij} g^{lm} (\bar{\nabla}_j \bar{R}_{0lim} - \bar{\nabla}_l \bar{R}_{0imj}) = g^{ij} g^{lm} \bar{\nabla}_j \bar{R}_{0lim} - g^{lm} g^{ij} \bar{\nabla}_l \bar{R}_{0imj} = 0.
\end{aligned}
$$

Thus it follows that

$$
h^{ij} (\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l) = \tilde{h}^{ij} (\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l),
$$

which is according to Cauchy-Schwarz less than or equal to

$$
|\tilde{h}^{ij}| |\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l| \leq \frac{1}{2} |\tilde{h}^{ij}|^2 + \frac{1}{2} |\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l|^2.
$$

Using $|\tilde{h}^{ij}|^2 = |\tilde{h}_{ij}|^2 = |A|^2 - \frac{H^2}{n} = f_\sigma H^\alpha$ and $|\bar{\nabla} \bar{R}|^2 \leq L^2$ we conclude

$$
-\frac{2}{H^\alpha} h^{ij} (\bar{\nabla}_j \bar{R}_{0li}{}^l + \bar{\nabla}_l \bar{R}_{0ij}{}^l) \leq C f_\sigma + C \frac{1}{H^\alpha}, \tag{48}
$$

where $C \geq 0$ depends on L.

Finally it is obvious that term [2] may be estimated as follows:

$$
(2 - \alpha)(|A|^2 + \bar{R}_{0l0}{}^l) f_\sigma \leq \sigma |A|^2 f_\sigma + C f_\sigma, \tag{49}
$$

where C depends on n, K_1 and K_2, the bounds of the ambient sectional curvatures.

Combining estimates (45),(47), (48) and (49) with (44) we get

$$
\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2(\alpha - 1)}{H} \langle \nabla H, \nabla f_\sigma \rangle - \frac{\varepsilon^2}{2H^\alpha} |\nabla H|^2 + \sigma |A|^2 f_\sigma + C \frac{1}{H^\alpha} + C f_\sigma. \tag{50}
$$

The constant C now depends on n, ε, K_1, K_2 and L.

Remark 5.9. Note that this improved inequality comprises three important pieces of information: The evolution equation of f_σ on the one hand, and, on the other hand, the fact that (modified) convexity is preserved according to theorem 4.14, as well as the fact that the ambient curvature terms are assumed to be bounded.

The following section will show how one can use it to derive a bound for high L^p-norms of f_σ.

5.4 Bounding high L^p-norms of f_σ

The next step towards a uniform upper bound of f_σ, which is our main goal, is to derive an upper bound C for all

$$||f_\sigma||_p = \left(\int_{M_t} f_\sigma^p \, d\mu_t\right)^{\frac{1}{p}}, \quad p \geq p_0, \, \sigma \leq \sigma(p),$$

independently of p and t. Here the constant C should only depend on M_0, $H_{\min}(0)$, as well as K_1, K_2 and L. Once we have established that, this knowledge strongly suggests that f_σ is indeed uniformly bounded from above for some sufficiently small σ. This is presented in detail in the next section, which will finally complete the proof.

We take inequality (50) as a starting point, which we integrate and then modify by a variety of estimations.

When integrating, note that the induced measure $d\mu = \mu_t(x)dx = \sqrt{\det g_{ij}}dx$ on M_t depends on t. When considering time derivatives of integrals over M_t, we therefore have to take into account that μ_t changes with t according to the following

Lemma 5.10. *The changing rate of the induced measure on M_t is given by*

$$\frac{\partial}{\partial t}\mu_t = -H^2 \cdot \mu_t.$$

Proof. We have the Radon-Nikodym derivative $d\mu/dx = \sqrt{\det g_{ij}}$.
Thus $\frac{\partial}{\partial t}\mu_t = \frac{\partial}{\partial t}\left(\sqrt{\det g_{ij}}\right)$, and if we work in an orthonormal frame we get

$$\frac{\partial}{\partial t}\left(\sqrt{\det g_{ij}}\right) = \det g_{ij} \sum_{i,j} g^{ij} \frac{\partial}{\partial t} g_{ij} = -H^2,$$

due to $\det(g_{ij}) = 1$, $g^{ij} = \delta^{ij}$ and lemma 3.5. The assertion follows. □

Remark 5.11. It follows $|M_0| \geq |M_t|$ for all $t \in [0, T[$, because of $\frac{\partial}{\partial t}|M_t| = -\int_{M_t} H^2 \, d\mu \leq 0$ on $[0, T[$.

By the above lemma we compute time derivatives of integrals as follows.

$$\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p \, d\mu = \int_{M_t} \frac{\partial}{\partial t}(f_\sigma^p) \, d\mu - \int_{M_t} H^2 f_\sigma^p \, d\mu = \int_{M_t} p f_\sigma^{p-1} \frac{\partial}{\partial t} f_\sigma \, d\mu - \int_{M_t} H^2 f_\sigma^p \, d\mu \tag{51}$$

In view of the first term of the right hand side of (51) we multiply (50) by $p f_\sigma^{p-1}$ and integrate. We then apply (51) to the resulting inequality. Moreover we use

$$\int_{M_t} f_\sigma^{p-1} \Delta f_\sigma \, d\mu = -(p-1) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu,$$

which is integration by parts (recall that M_t has no boundary), and

$$\int_{M_t} H^{-1} \langle \nabla H, \nabla f_\sigma \rangle f_\sigma^{p-1}\, d\mu \leq \int_{M_t} H^{-1} |\nabla H| |\nabla f_\sigma| f_\sigma^{p-1}\, d\mu,$$

which follows from the Cauchy-Schwarz inequality. Thus we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p\, d\mu + \underbrace{p(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\, d\mu}_{[1]} + \underbrace{\frac{p\varepsilon^2}{2}\int_{M_t}\frac{1}{H^\alpha} f_\sigma^{p-1}|\nabla H|^2\, d\mu}_{[2]} \\
+ &\int_{M_t} H^2 f_\sigma^p\, d\mu \leq \underbrace{2(\alpha-1)p\int_{M_t} H^{-1}|\nabla H||\nabla f_\sigma| f_\sigma^{p-1}\, d\mu}_{[3]} + \sigma p \int_{M_t} |A|^2 f_\sigma^p\, d\mu \\
+ &\; pC\int_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}\, d\mu + pC\int_{M_t} f_\sigma^p\, d\mu, \qquad (52)
\end{aligned}$$

where C depends, as in (50), on n, ε, K_1, K_2 and L.

Let us consider the terms involved in (52). We follow the idea of making certain terms of the inequality get "absorbed" by other terms by treating them with standard inequalities.

The terms [1], [2] and [3] of (52), for example, suggest to be "merged" by Chauchy's inequality with η given by

$$ab \leq \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2 \qquad (53)$$

for any $\eta > 0$, $a, b \in \mathbb{R}$.

Let us suppose $p \geq 2$ from now on.

Then, setting $a := |\nabla f_\sigma|$, $b := f|\nabla H|/H$ and $\eta := (p-1)/2$ in (53), we get

$$2p\int_{M_t} H^{-1}|\nabla H||\nabla f_\sigma| f_\sigma^{p-1}\, d\mu \leq \frac{1}{2}p(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\, d\mu + 2\frac{p}{p-1}\int_{M_t}\frac{1}{H^2} f_\sigma^p|\nabla H|^2\, d\mu.$$

Combining this with $2(\alpha-1) \leq 2$ and $f_\sigma \leq H^{2-\alpha}$ (follows from remark 5.3), we conclude that term [3] is smaller than or equal to

$$\frac{1}{2}p(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\, d\mu + 2\frac{p}{p-1}\int_{M_t}\frac{1}{H^\alpha} f_\sigma^{p-1}|\nabla H|^2\, d\mu. \qquad (54)$$

Note that basically these are the terms [1] and [2]. Applying this estimate to (52) and afterwards subtracting (54) from both sides, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{M_t} f_\sigma^p\, d\mu + \frac{1}{2}p(p-1)\underbrace{\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\, d\mu}_{[1]} + \left(\frac{p\varepsilon^2}{2} - \frac{2p}{p-1}\right)\underbrace{\int_{M_t}\frac{1}{H^\alpha} f_\sigma^{p-1}|\nabla H|^2\, d\mu}_{[2]} \\
\leq &\; \sigma p\underbrace{\int_{M_t} H^2 f_\sigma^p\, d\mu}_{[3]} + \underbrace{pC\int_{M_t}\frac{f_\sigma^{p-1}}{H^\alpha}\, d\mu + pC\int_{M_t} f_\sigma^p\, d\mu}_{[4]}. \qquad (55)
\end{aligned}$$

As it will turn out in view of the following estimations it is handy to demand that

$$\frac{p\varepsilon^2}{2} - \frac{2p}{p-1} \geq \frac{p\varepsilon^2}{4},$$

which is equivalent to $p \geq (8+\varepsilon^2)\varepsilon^{-2}$ and may, for example, be simplified to the condition

$$p \geq 200\varepsilon^{-2}, \tag{56}$$

since we chose ε to be small. Let p satisfy this condition from now on.

Remark 5.12. Some more work has to be done in order to further simplify inequality (55). Consider terms [1], [2] and [3]. Unfortunately term [3] including its coefficient is positive and thus cannot simply be omitted. On the other hand, for our choice of p terms [1] and [2] are positive, too, which gives hope that they might cancel out with term [3]. This is suggested even more, as σ may be chosen arbitrarily small. In fact, this estimation is possible, as the following lemma shows. However, its proof is rather extensive as many estimates are involved. We therefore have decided to only describe the crucial ideas here and give the thorough proof in the appendix.

Lemma 5.13. *Let $p \geq 2$. Then for any $\eta > 0$ and any $0 \leq \sigma \leq \frac{1}{2}$ we have the estimate*

$$\begin{aligned} \frac{n\varepsilon^2}{2}\int\limits_{M_t} H^2 f_\sigma^p \, d\mu \;&\leq\; (\eta(p-1)+4)\int\limits_{M_t} \frac{1}{H^\alpha} f_\sigma^{p-1}|\nabla H|^2 \, d\mu + \frac{p-1}{\eta}\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \, d\mu \\ &+\; C\int\limits_{M_t} H^2 \, d\mu + C\int\limits_{M_t} f_\sigma^p \, d\mu + C, \end{aligned} \tag{57}$$

where C depends on ε, M_0, $H_{\min}(0)$, K_1, K_2, and L.

Scetch of proof. The main idea is to exploit an inequality for the Laplacian derived from (43). We integrate it by parts over M_t and take advantage of the fact that there are no boundary terms occurring. Then the remaining terms are estimated by means of the Chauchy-Schwarz inequality and Chauchy's inequality with η (see (53)). The lemma follows by applying relation (35) once more in order to estimate terms involving h_{ij}.

We multiply (57) by $2\sigma p/(n\varepsilon^2)$ and apply it to term [3] of (55). In the resulting inequality we then combine identical integrals and obtain the relation

$$\begin{aligned} &\frac{\partial}{\partial t}\int\limits_{M_t} f_\sigma^p \, d\mu + \underbrace{\left[\frac{1}{2}p(p-1) - \frac{2\sigma p}{n\varepsilon^2}\frac{p-1}{\eta}\right]}_{:=\xi_1}\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 \, d\mu \\ +\;& \underbrace{\left[\frac{p\varepsilon^2}{4} - \frac{2\sigma p}{n\varepsilon^2}(\eta(p-1)+4)\right]}_{:=\xi_2}\int\limits_{M_t} \frac{1}{H^\alpha} f_\sigma^{p-1}|\nabla H|^2 \, d\mu \\ \leq\;& pC\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha} \, d\mu + pC\int\limits_{M_t} f_\sigma^p \, d\mu + \frac{2\sigma p}{n\varepsilon^2}\left(C\int\limits_{M_t} H^2 \, d\mu + C\int\limits_{M_t} f_\sigma^p \, d\mu + C\right). \end{aligned} \tag{58}$$

Clearly we may now choose $\sigma = \sigma(p) > 0$ so small that - for some suitable choice of $\eta = \eta(p) > 0$ - the expressions ξ_1 and ξ_2 are not negative. We then may modify the estimate by setting $\xi_1 = \xi_2 = 0$. Observe that the same holds for all $0 < \tilde{\sigma} < \sigma$ with the same η. We may, for instance, choose

$$\sigma \leq \frac{n\varepsilon^3}{32\sqrt{p}},$$

along with some suitable η, as one readily verifies.

The terms on the right hand side of (58) may also be estimated. Since H is bounded from below by $H_{\min}(0) > 0$ for all $0 \leq t < T$, we have

$$\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha} \, d\mu \leq \frac{1}{(H_{\min}(0))^\alpha} \int\limits_{M_t} f_\sigma^{p-1} \, d\mu = C(H_{\min}(0)) \int\limits_{M_t} f_\sigma^{p-1} \, d\mu.$$

We further estimate by remark 5.11

$$\begin{aligned} \int\limits_{M_t} f_\sigma^{p-1} \, d\mu &\leq |M_t| + \int\limits_{M_t} f_\sigma^p \, d\mu \\ &\leq |M_0| + \int\limits_{M_t} f_\sigma^p \, d\mu = C(M_0)^p + \int\limits_{M_t} f_\sigma^p \, d\mu. \end{aligned}$$

By means of these two estimates (58) takes the final form

$$\frac{\partial}{\partial t} \int\limits_{M_t} f_\sigma^p \, d\mu \leq pC \int\limits_{M_t} f_\sigma^p \, d\mu + pC^p \int\limits_{M_t} H^2 \, d\mu + pC^p, \tag{59}$$

which holds for all $p \geq 200\varepsilon^{-2}$ and $\sigma \leq n\varepsilon^3 p^{1/2} 32^{-1}$.
Here we have $C = C(n, \varepsilon, K_1, K_2, L, M_0, H_{\min}(0))$.

This inequality can now be used to derive an upper bound for $||f_\sigma||_p$ by means of the following lemma.

Lemma 5.14. *Suppose that there is some $C > 0$ and some $p_0 > 0$ and $\sigma_0 > 0$ such that for all $p \geq p_0$ there is a sufficiently small $0 < \sigma(p) < \sigma_0$ such that*

$$\frac{\partial}{\partial t} \int\limits_{M_t} f_{\sigma(p)}^p \, d\mu \leq pC \int\limits_{M_t} f_{\sigma(p)}^p \, d\mu + pC^p \int\limits_{M_t} H^2 \, d\mu + pC^p \quad \text{on } [0, T[. \tag{60}$$

Then there is some $\infty > K = K(C, M_0) > 0$ such that

$$\left(\int_{M_t} f_{\sigma(p)}^p \, d\mu \right)^{\frac{1}{p}} \leq K \quad \text{for all } p \geq p_0 \text{ and for all } t \in [0, T[.$$

Proof. First observe that due to lemma 5.10 we have

$$\int\limits_{M_t} H^2 \, d\mu = -\frac{\partial}{\partial t} \int\limits_{M_t} d\mu = -\frac{\partial}{\partial t} |M_t|.$$

Let now $p \geq p_0$. Since $e^{pCt} \geq 1$ for $t \in [0, T[$, it follows for $\varphi_p(t) := \int_{M_t} f^p_{\sigma(p)}\, d\mu$ that we have

$$\frac{\partial}{\partial t}\varphi_p(t) \leq pC\varphi_p(t) + pC^p \int\limits_{M_t} H^2\, d\mu + e^{pCt} pC^p.$$

Let $\tilde{\varphi}_p$ be a solution of this *equality* with $\tilde{\varphi}_p(0) = \varphi_p(0)$. Then it is $\varphi_p(t) \leq \tilde{\varphi}_p(t)$ on $[0, T[$. The function $\tilde{\varphi}_p$, however, is given by $\tilde{\varphi}_p(t) = \varphi_p(0) + tpC^p e^{pCt} + pC^p(|M_0| - |M_t|)$. Thus

$$\sup_{[0,T[} \varphi_p(t) \leq \varphi_p(0) + TpC^p e^{pCT} + pC^p|M_0|.$$

Note that $\sqrt[p]{p} \to 1$ for $p \to \infty$. For all $t \in [0, T[$ we then conclude

$$\left(\int_{M_t} f^p_{\sigma(p)}\, d\mu\right)^{\frac{1}{p}} \leq (\sup_{M_0} f_{\sigma_0} + TpC^p e^{pCT} + pC^p|M_0|)^{\frac{1}{p}} \leq K,$$

which holds for all $p \geq p_0$ and for some suitable $K = K(C, M_0) > 0$, since M_0 is compact and $T < \infty$. □

Remark 5.15. Observe that here we have explicitly exploited the fact that in the current setting the mean curvature flow can have a solution only on a finite time interval $[0, T[$, $T < \infty$.

In particular we obtain the following result.

Theorem 5.16. *For all*

$$p \geq \frac{200}{\varepsilon^2} \quad \text{and} \quad \sigma \leq \frac{n\varepsilon^3}{32\sqrt{p}}$$

we have $||f_\sigma||_p \leq K$ on the whole interval $0 \leq t < T$. The constant K depends on n, ε, M_0, $H_{\min}(0)$ and the bounds K_1, K_2 and L of the ambient curvature.

As a direct consequence of this theorem we get

Corollary 5.17. *Let $m \in \mathbb{N}$. Then there is some $K > 0$ depending on the same quantities as above such that the relation*

$$\left(\int_{M_t} H^m f^p_\sigma\, d\mu\right)^{\frac{1}{p}} \leq K \quad \text{holds for all } t \in [0, T[\text{ and } p \geq p_0,$$

provided that we choose p_0 large enough and $\sigma := \sigma(p) > 0$ small enough.

Proof. In view of the definition $f_\sigma := (|A|^2 - H^2/n)/H^{2-\sigma}$ we see that we always may arrange that

$$\left(\int_{M_t} H^m f^p_\sigma\, d\mu\right)^{\frac{1}{p}} = \left(\int_{M_t} f^p_{\sigma'}\, d\mu\right)^{\frac{1}{p}}$$

for $\sigma' = \sigma + m/p$. If p is large enough, σ' remains small enough such that theorem 5.16 applies. □

The above theorem and its corollary are the crucial means which we employ to derive the bound for f_σ in the next section.

5.5 Bounding f_σ

In this section we will finally derive a uniform bound for f_σ for some $\sigma > 0$. We roughly describe the basic idea: f_σ is bounded uniformly in t if there is some $k \in \mathbb{N}$ such that $\max\{f_\sigma - k, 0\}$ remains zero everywhere on M_t and for all $t \in [0, T[$. The number k may be arbitrarily large - as long as it does exist.

The idea the procedure is based upon is inspired by the following lemma on certain non-increasing positive functions. Consider the sets $A(c) := \{x \in M_t \mid f_\sigma(x) > c\}$ for fixed $0 \leq t < T$ and $c \in \mathbb{R}_+$. Obviously these sets have the property $A(d) \subseteq A(c)$ whenever $d \geq c$. Accordingly, the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(c) := ||A(c)|| := \int_0^T \int_{A(c)} d\mu \, dt \tag{61}$$

is non-increasing and positive.

It is evident that the existence of the desired uniform bound of f_σ is equivalent to the fact that $A(k) = \emptyset$ for some $k \in \mathbb{N}$ and for all $0 \leq t < T$, which, in turn, is the same as demanding that $\varphi(c)$ vanish for all $c > k$.

It thus remains to show that φ behaves like this. Since φ is non-increasing and positive, it meets the requirements of the following lemma, which reduces the problem to the derivation of a certain condition of decrease for φ.

Lemma 5.18. *Let φ be a non-decreasing, non-negative function defined on $[k_0, \infty[$ such that for all $h > k \geq k_0$ the condition of decrease*

$$\varphi(h)(h-k)^p \leq C(\varphi(k))^\gamma \quad \textit{for some positive constants } p,\ C,\ \gamma \tag{62}$$

is satisfied with $\gamma > 1$. Then $\varphi(x)$ vanishes for all $x \geq k_0 + d$, where

$$d^p = C \cdot 2^{\frac{p\gamma}{\gamma-1}} (\varphi(k_0))^{\gamma-1}.$$

Proof. Consider the sequence $k_s := k_0 + d - \frac{d}{2^s}$. Obviously it tends to $k_0 + d$ from below and we have finished if we manage to prove that $\varphi(k_s)$ tends to zero for $s \to \infty$.

Using induction on s, we achieve this by showing that

$$\varphi(k_s) \leq \frac{\varphi(k_0)}{2^{-s\mu}} \quad \text{with} \quad \mu = \frac{p}{1-\gamma} \quad \text{for all } s \in \mathbb{N}. \tag{63}$$

The case $s = 0$ is trivial.

Now suppose that (63) is satisfied for s, and we want to show that it also holds for $s + 1$.

We have

$$(k_{s+1} - k_s)^p = -\frac{d}{2^{s+1}} + \frac{d}{2^s} = \frac{d}{2^{s+1}}$$

and we thus observe that by assumption (62) we have

$$\varphi(k_{s+1}) \leq C \frac{2^{p(s+1)}}{d^p} (\varphi(k_s))^\gamma.$$

From this it follows by the induction hypothesis that

$$\varphi(k_{s+1}) \leq C \frac{2^{p(s+1)}}{d^p} \left(\frac{\varphi(k_0)}{2^{-s\mu}} \right)^{\gamma},$$

and according to the definition of d^p we obtain

$$\varphi(k_{s+1}) < \frac{\varphi(k_0)}{2^{-(s+1)\mu}}.$$

□

In the sequel we will demonstrate how condition (62) can be verified for the particular φ defined above: We will gradually derive it from an integral inequality which we partly already have established. The main idea all estimations are inspired by is the statement of corollary 5.17, saying that integrals of terms of the form $H^m f_\sigma^p$ are bounded *uniformly in* t.

Beforehand we provide a Sobolev inequality for general submanifolds of Riemannian manifolds, which we do not prove, for this would go beyond the scope of this work.

Remark 5.19. We need this inequality since we will have to control gradient terms occurring under the integral. In such situations one usually employs Sobolev-type estimates in order to relate the gradient integral to an integral involving the function itself. In contrast to Euclidean space, however, such a Sobolev inequality is not immediately available for the present setting of general Riemannian manifolds. D. Hoffman and J. Spruck (cf. [9]) have established a general Sobolev inequality holding on arbitrary submanifolds of Riemannian manifolds. Depending on the size of the support of the function v to be treated by the inequality, though, the ambient space has to behave "well" in that the *injectivity radius* of the ambient space, which is the maximum distance below which geodesics are minimizing, must be bounded by an arbitrarily small, strictly positive quantity.

Definition 5.20. Let $M \subset N$ be a submanifold of a (complete) Riemannian manifold N, and let $p \in M$. Furthermore let $\gamma : [0, \infty[\to N$ be a normalized geodesic in N with $\gamma(0) = p$.
By the *cut point of* p *along* γ we denote the point $\gamma(t)$, where $t > 0$ is the minimal number such that $\gamma([0, t])$ is *not* minimizing.

By the *cut locus of* p $C_m(p)$ we denote the union of all cut points of p.

Finally we denote by the *injectivity radius of* N *restricted to* M the number $i|_M(N) := \inf_{p \in M} d(p, C_m(p))$.

We cite theorem 2.1 from [9], which we have adapted to our case. It then takes the form

Theorem 5.21. *Let* $n \geq 2$. *Further let* $M^n \to N^{n+1}$ *be an isometric hypersurface immersion of Riemannian manifolds. Assume that* M *has no boundary and that the sectional curvature* $K_p(\sigma)$ *be bounded from above by* K_2, $K_2 > 0$, *for all* $p \in N^{n+1}$ *and all two-planes* $\sigma \subset T_p N^{n+1}$. *Furthermore let* v *be a nonnegative Lipschitz function on* M. *Then*

$$\left(\int_M v^{\frac{n}{n-1}} \, d\mu \right)^{\frac{n-1}{n}} \leq C(n) \left(\int_M |\nabla v| \, d\mu + \int_M |H| v \, d\mu \right), \tag{64}$$

provided

$$K_2(1-\alpha)^{-\frac{2}{n}}(\omega_n^{-1}|supp\ v|)^{\frac{2}{n}} \leq 1 \quad \textit{and} \quad 2\rho_0 \leq i|_M(N),$$

where ω_n *is the volume of the unit ball and*

$$\rho_0 = K_2^{-1/2} \arcsin(K_2^{1/2}(1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}|supp\ v|)^{\frac{1}{n}}).$$

Here α *is a free parameter,* $0 < \alpha < 1$*, and*

$$C(n) = \frac{1}{2}\pi \cdot 2^{n-2}\alpha^{-1}(1-\alpha)^{-\frac{1}{n}}\frac{n}{n-1}\omega_n^{-\frac{1}{n}}.$$

Remark 5.22. Regarding the somewhat complicated conditions, we note that in our case the particular values of ρ_0, α and $C(n)$ will not matter; it is only important to observe that whenever $K_2 > 0$ is given along with a lower bound $\chi > 0$ for $i|_M(N)$, the theorem holds if we restrict ourselves to functions v whose support supp v is small enough, dependent on K_2 and χ. Of course the upper bound K_2 of the ambient sectional curvature coincides with the K_2 introduced earlier and is thus already assumed to exist. However, we have to additionally assume the existence of the lower bound χ for $i(N)$ to make the theorem apply.

Remark 5.23. We will not explicitly need the Sobolev inequality in the form given above. Rather we modify the exponents in a typical way: Let $n \geq 3$. For $\xi := \frac{2n-2}{n-2} > 1$ we apply (64) on v^ξ and get by the Hölder inequality

$$\begin{aligned}
\left(\int_M v^{\frac{\xi n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}} &\leq C(n)\xi \int_M v^{\xi-1}|\nabla v|\,d\mu + C(n)\int_M Hv^\xi\,d\mu \\
&\leq C(n)\xi \left(\int_M v^{2(\xi-1)}\,d\mu\right)^{\frac{1}{2}} \left(\int_M |\nabla v|^2\,d\mu\right)^{\frac{1}{2}} \\
&+ C(n)\left(\int_M v^{2(\xi-1)}\,d\mu\right)^{\frac{1}{2}} \left(\int_M H^2v^2\,d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

The reason for the particular choice of ξ is that we have $\xi n/(n-1) = 2(\xi-1) = 2n/(n-2)$ and $(n-1)/n - 1/2 = (n-2)/2n$. Division by $(\int_M v^{2(\xi-1)})^{1/2}$ yields

$$\left(\int_M v^{\frac{2n}{n-2}}\,d\mu\right)^{\frac{n-2}{2n}} \leq \tilde{C}(n)\left(\int_M |\nabla v|^2\,d\mu\right)^{\frac{1}{2}} + \tilde{C}(n)\left(\int_M H^2v^2\,d\mu\right)^{\frac{1}{2}}, \tag{65}$$

with $\tilde{C}(n) = \xi C(n)$. We will need this version of the Soboley inequality later.

Additionally note that in the case $n = 2$ we obtain a similar alternative from (64): This is just (65) with the exponents $2n/(n-2)$ and $(n-2)/2n$ replaced by q and $1/q$, respectively, where q may take any value $1 \leq q < \infty$.

We now derive the condition needed in lemma 5.18. We start by defining the set of functions to work with:

$$f_{\sigma,k} := \max\{f_\sigma - k, 0\} \quad \text{for} \quad k \geq k_0 := \sup_{M_0} f_\sigma.$$

Remark 5.24. The definition of k_0 implies $f_{\sigma,k} \equiv 0$ on M_0, which will be of importance later.

In section 5.3 we have derived relation (50), which has the form

$$\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2(\alpha-1)}{H}\langle \nabla H, \nabla f_\sigma \rangle - \frac{\varepsilon^2}{2H^\alpha}|\nabla H|^2 + \sigma |A|^2 f_\sigma + C\frac{1}{H^\alpha} + C f_\sigma,$$

where $C = C(n, \varepsilon, K_1, K_2, L)$. Recall that this was a consequence of the evolution equation for f_σ and theorem 4.14. We use this inequality as a starting point and proceed similarly to how we did in section 5.4: First we multiply it by $p f_{\sigma,k}^{p-1}$. In section 5.4 we then integrated over M_t; in this case we instead integrate over $A(k)$, since $f_{\sigma,k}$ vanishes outside $A(k)$. Taking into account that

$$\frac{\partial}{\partial t} f_\sigma = \frac{\partial}{\partial t} f_{\sigma,k} \quad \text{and} \quad \nabla f_\sigma = \nabla f_{\sigma,k} \quad \text{on } A(k)$$

and $f_{\sigma,k} \equiv 0$ on $\partial A(k)$, we conclude analogously to (55) and get

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{A(k)} f_{\sigma,k}^p \, d\mu + \frac{1}{2} p(p-1) \int_{A(k)} f_{\sigma,k}^{p-2} |\nabla f_{\sigma,k}|^2 \, d\mu \\ \leq \;& \sigma p \int_{A(k)} H^2 f_\sigma^p \, d\mu + pC \int_{A(k)} \frac{f_{\sigma,k}^{p-1}}{H^\alpha} \, d\mu + pC \int_{A(k)} f_{\sigma,k}^p \, d\mu. \end{aligned} \tag{66}$$

Remark 5.25. Again, $C = C(n, \varepsilon, K_1, K_2, L)$. Moreover, we can control the two last terms on the right hand side by a constant $\tilde{C}(p)$ depending on p, C, M_0 and $H_{\min}(0)$, by taking into account that $H^\alpha \geq (H_{\min}(0))^\alpha$, $\int_{A(k)} f_{\sigma,k}^{p-1} \, d\mu \leq |M_t| + \int_{A(k)} f_{\sigma,k}^p \, d\mu$, as well as $|M_t| \leq |M_0|$. Plus, $\tilde{C}(p)$ and thus the whole left hand side can be estimated from above by

$$C(p) \int_{A(k)} H^2 f_\sigma^p \, d\mu,$$

since we have $f_\sigma^p \geq k^p$ on $A(k)$ and $H^2 \geq (H_{\min}(0))^2$. Here $C(p)$ depends on p, n, ε, M_0, $H_{\min}(0)$ and the bounds of the ambient curvature.

It is easy to verify that on $A(k)$ we have

$$\frac{1}{2} p(p-1) f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 \leq |\nabla f_{\sigma,k}^{p/2}|^2,$$

which - applied to (66) - yields with $v := f_{\sigma,k}^{p/2}$ and remark 5.25 the estimate

$$\frac{\partial}{\partial t} \int_{A(k)} v^2 \, d\mu + \int_{A(k)} |\nabla v|^2 \, d\mu \leq C(p) \int_{A(k)} H^2 f_\sigma^p \, d\mu. \tag{67}$$

Remark 5.26. Observe that the term on the right hand side suggests to be treated by corollary 5.17, whereas the first term is the one we want to control. We thus have to take care for the term on the left hand side involving ∇v, which can now be treated by the adapted version of the Sobolev inequality (65) we have provided above. In view of remark 5.22 we have to ensure that the support of v is sufficiently small, where "sufficiently" refers to the given upper bound $K_2 > 0$ of the ambient sectional curvature and $\chi > 0$, the lower bound of the injectivity radius of the ambient space.

From now on we will assume the injectivity radius of N to be uniformly bounded from below by $\chi > 0$.

We cope with the restriction imposed by the injectivity radius by observing that supp v gets arbitrarily small for large k according to

$$|\text{supp } v| \leq |A(k)| = \int_{A(k)} d\mu \leq \frac{1}{k} \int_{M_t} f_\sigma \, d\mu,$$

which, in view of theorem 5.16, is bounded from above by C/k, where C depends on n, ε, M_0, $H_{\min}(0)$ and the ambient curvature bounds.

Thus $|\text{supp } v|$ gets arbitrarily small if we choose k large enough. Consequently we restrict our considerations to all $k \geq k_1$, where $k_1 = k_1(\chi) \geq k_0$ is large enough such that (65) is valid.

Now we are ready to apply the Sobolev inequality. Squaring (65) and afterwards applying the relation $(a+b)^2 \leq 2a^2 + 2b^2$ (which holds for all $a, b \in \mathbb{R}$) to the right hand side, we obtain for a suitable new choice of $C(n)$

$$\left(\int_{M_t} v^{2q} \, d\mu \right)^{\frac{1}{q}} \leq C(n) \int_{M_t} |\nabla v|^2 \, d\mu + C(n) \int_{M_t} v^{2q} \, d\mu.$$

By the Hölder inequality we conclude from (65) that

$$\left(\int_{M_t} v^{2q} \, d\mu \right)^{\frac{1}{q}} \leq C(n) \int_{M_t} |\nabla v|^2 \, d\mu + C(n) \left(\int_{\text{supp } v} H^n \, d\mu \right)^{\frac{2}{n}} \left(\int_{M_t} v^{2q} \, d\mu \right)^{\frac{1}{q}},$$

where here and in the last inequality we have set

$$q = \frac{n}{n-2} \quad \text{for } n > 2 \text{ and } \quad q < \infty \quad \text{for } n = 2,$$

according to remark 5.22. This relation shows that we indeed are able to handle the gradient term by estimating it by integrals of (powers of) v.

The additional term involving H^n, which distinguishes the general Sobolev inequality from the well-known one in $\mathbb{R}^n$, is not very much of importance here, as it can be made arbitrarily small by (once more) increasing k: As supp $v \subseteq A(k)$ and $f_\sigma/k > 1$ on $A(k)$ we get by Corollary 5.17

$$\left(\int_{\text{supp } v} H^n \, d\mu \right)^{\frac{2}{n}} \leq \left(\frac{1}{k^p} \right)^{\frac{2}{n}} \left(\int_{A(k)} H^n f_\sigma^p \, d\mu \right)^{\frac{2}{n}} \leq \left(\frac{1}{k^p} \right)^{\frac{2}{n}} C^{\frac{2p}{n}}$$

provided that p is large enough and σ is small enough.

We therefore raise the considered k, if necessary, that is we introduce a new lower bound $k_2 \geq k_1$ and restrict ourselves to all $k \geq k_2$. Then it follows from (67)

$$\frac{\partial}{\partial t} \int_{A(k)} v^2 \, d\mu + \tilde{C}(n) \left(\int_{A(k)} v^{2q} \, d\mu \right)^{\frac{1}{q}} \leq C(p) \int_{A(k)} H^2 f_\sigma^p \, d\mu, \tag{68}$$

where again $\tilde{C}(n) > 0$ only depends on n. Now we consider $t_0 \in [0, T]$ such that

$$\int_{A(k)} v^2 \, d\mu \,|_{t_0} \geq \int_{A(k)} v^2 \, d\mu \,|_t \text{ for all } t \in [0, T].$$

Moreover, observe that due to our choice of k_0 (remark 5.24) we have

$$\int_{A(k)} v^2 \, d\mu \,|_0 = 0 \quad \text{for all } k \geq k_0.$$

Thus

$$\sup_{[0,T]} \int_{A(k)} v^2 \, d\mu = \int_0^{t_0} \frac{\partial}{\partial t} \left(\int_{A(k)} v^2 \, d\mu \right) dt \leq \int_0^T \frac{\partial}{\partial t} \left(\int_{A(k)} v^2 \, d\mu \right) dt,$$

which yields in combination with (68)

$$\sup_{[0,T]} \int_{A(k)} v^2 \, d\mu + \tilde{C}(n) \int_0^T \left(\int_{A(k)} v^{2q} \, d\mu \right)^{\frac{1}{q}} dt \leq C(p) \int_0^T \int_{A(k)} H^2 f_\sigma^p \, d\mu \, dt. \tag{69}$$

In order to proceed we have to invoke the interpolation inequality for L^p-spaces (see Evans [4]), which is a direct implication of the Hölder inequality. In our case it takes the form

$$\left(\int_{A(k)} v^{2q_0} \, d\mu \right)^{1/q_0} \leq \left(\int_{A(k)} v^{2q} \, d\mu \right)^{a/q} \left(\int_{A(k)} v^2 \, d\mu \right)^{(1-a)},$$

for $q_0 > 1$, and $a > 0$ has to be chosen such that $1/q_0 = a/q + (1-a)$. Here we set $a = 1/q_0$, which implies $1 < q_0 < q$ due to the value of q. (In the case $n = 2$ we simply choose q large enough.) Applying the interpolation inequality to (69) yields the estimate

$$\left(\int_0^T \int_{A(k)} v^{2q_0} \, d\mu \, dt \right)^{1/q_0} \leq C(p) \int_0^T \int_{A(k)} H^2 f_\sigma^p \, d\mu \, dt \tag{70}$$

with $C(p) = C(p, n, \varepsilon, M_0, H_{\min}(0), K_1, K_2, L)$. Employing the Hölder inequality once more yields that this is less than or equal to

$$C(p)|A(k)|^{1-1/r} \left(\int_0^T \int_{A(k)} H^{2r} f_\sigma^{pr} \, d\mu \, dt \right)^{1/r}, \quad \text{for some } r > 1 \text{ to be chosen.}$$

The power q_0 of the left integral in (70) can be avoided by means of the Hölder inequality. For we have

$$\int_0^T \int_{A(k)} v^2 \, d\mu \, dt \leq |A(k)|^{1-1/q_0} \left(\int_0^T \int_{A(k)} v^{2q_0} \, d\mu \, dt \right)^{1/q_0},$$

whence by $f_{\sigma,k} = v^2$ we "shift" the power q_0 to the right hand side to finally find

$$\int_0^T \int_{A(k)} f_{\sigma,k}^p \, d\mu \, dt \leq C(p)|A(k)|^{2-1/q_0-1/r} \left(\int_0^T \int_{A(k)} H^{2r} f_\sigma^{pr} \, d\mu \, dt \right)^{1/r}.$$

Now we have reached our aim. For observe that by definition of $f_{\sigma,k}$ we can reason as follows: Let $h > k$. Then for all $x \in A(h) \subset A(k)$ we have $f_{\sigma,k}(x) \geq h - k$. Thus we may estimate the above left hand side from below by $|h-k|^p |A(h)|$.

On the other hand, the integral on the above right hand side is of such a form that corollary 5.17 applies *whatever value* $r > 1$ *may take*, provided that p is large and σ is small enough. *Let us accordingly fix the values of these two variables in such a way that the* *corollary applies.* (These values depend on r). Then the entire integral may be replaced by a constant.

Note that if we choose $r > 1$ such that $2 - 1/q_0 - 1/r = \gamma > 1$, we exactly have derived the condition required by lemma 5.18. Thus this lemma applies and ensures that indeed there is some $\sigma > 0$ such that f_σ is bounded uniformly in t.

Remark 5.27. Note that the lemma applies despite the fact that the constant $C(p)$ still depends on p as well as n, ε, M_0, $H_{\min}(0)$, K_1, K_2 and L.

We summarize:

Theorem 5.28. *Assume the conditions of theorem 4.14 to be satisfied. Moreover assume that*

$$sec(p) \leq K_2 \quad \text{for all } p \in N \text{ and some } K_2 \geq 0$$

and

$$i_p(N) \geq \chi > 0 \quad \text{for all } p \in N,$$

where $i_p(N)$ is the injectivity radius of N at p.

Then there is some $\sigma > 0$ such that $f_\sigma = \frac{|A|^2 - H^2/n}{H^{2-\sigma}}$ is bounded uniformly for all $t \in [0, T[$. That is, $|A|^2 - H^2/n$ becomes small compared to H^2 and the eigenvalues of h_{ij} get pinched together, if H tends to infinity.

We thus have finally shown that the principal curvatures get pinched together towards the end of the finite interval $[0, T[$ on which the mean curvature flow exists. This fact is an important step towards the main result in [8] stating that volume-preserving homothetic expansions of the evolving surface will converge to a sphere.

6 Example I: Hyperbolic space

The purpose of the following two chapters is to investigate the question posed in remark 4.15, namely if the convexity theorem 4.14 of chapter 4 might be improved by using sharper arguments in the proof. In particular, we concentrate on the terms involving the ambient curvature bounds K_1 and L, which we had to take into account in the modified convexity condition

$$Hh_{ij} > nK_1 g_{ij} + \frac{n^2}{H} L g_{ij}. \tag{71}$$

Thus the notion of convexity had to be adapted to the particular geometrical structure of the ambient space. Recall that the occurrence of these terms is due to our choice of the function f in the proof of theorem 4.14, and it is fair to assume that this choice might not be optimal. The following examples, however, suggest that we cannot expect a crucial improvement, for example by replacing the above modified convexity condition by

$$h_{ij} > 0. \tag{72}$$

Moreover we examine the consequences of omitting only one of the additional terms.

For this purpose we consider two different ambient spaces N_1 and N_2 such that in N_1 we have $K_1 > 0$ and $L = 0$, whereas in N_2 we have $K_1 = 0$ and $L > 0$. In each case we (at least locally) construct a smooth convex hypersurface M_0 losing its convexity as soon as it is moved by mean curvature flow. (By "convex" we now and in the sequel always refer to the original notion of convexity in the sense of definition 4.1.)

Remark 6.1. A special case occurs if we have Euclidean ambient space. Then $K_1 = L = 0$, and the modified convexity condition (71) is equivalent to (72).

In the present chapter we discuss the first example where we choose a locally symmetric ambient space N_1 with strictly negative sectional curvature, the hyperbolic space. Local symmetry is equivalent to $\bar{\nabla}\bar{R} \equiv 0$, whence we have $K_1 > 0$ and $L = 0$.

In the next chapter we choose N_2 to be a distorted sphere, the Berger sphere, which has positive sectional curvatures while not being locally symmetric. Hence $K_1 = 0$ and $L > 0$.

Both times we use the same approach: We construct a smooth hypersurface immersion M which is (not necessarily strictly) convex away from one point p. At p all eigenvalues κ_i of h_{ij} are constructed to be strictly positive, except for one of them, say κ_1, which we demand to satisfy $\kappa_1 = 0$.

We then invoke the evolution equation for the second fundamental form, which we use to show that h_{ij} must take a strictly negative eigenvalue at p as soon as the surface is moved by mean curvature flow.

We now discuss the first example.

6.1 Hyperbolic space and its geometry

It is easy to find a locally symmetric space (i.e. $\bar{\nabla}\bar{R} \equiv 0$) with strictly negative sectional curvature. For if the (sectional) curvature is *constant*, the space is automatically locally symmetric, due to the following

Lemma 6.2. *Spaces of constant sectional curvature are locally symmetric.*

Proof. Let M be a space of constant curvature. Suppose that we have constant sectional curvature at all $p \in M$ and all two-dimensional planes in T_pM. This implies that if we choose local coordinates $\{x^i\}$ around p we have for all i, j

$$\frac{R_{ijij}}{g_{ii}g_{jj} - (g_{ij})^2} = K.$$

It is now a purely algebraic fact that all components of R can be recovered by means of the relation above. In fact we have

$$R_{ijkl} = K[g_{ik}g_{jl} - g_{jk}g_{il}]$$

for all i, j, k, l.

From this we derive for the components of ∇R under the assumption that $g_{ij} = \delta_{ij}$ at T_pM:

$$\begin{aligned}
\nabla_m R_{ijkl} &= \frac{\partial}{\partial x^m} R_{ijkl} - \Gamma^n_{mi} R_{njkl} - \Gamma^n_{mj} R_{inkl} - \Gamma^n_{mk} R_{ijnl} - \Gamma^n_{ml} R_{ijkn} \\
&= K(\delta_{jl} \frac{\partial}{\partial x^m} g_{ik} + \delta_{ik} \frac{\partial}{\partial x^m} g_{jl} - \delta_{il} \frac{\partial}{\partial x^m} g_{jk} - \delta_{jk} \frac{\partial}{\partial x^m} g_{il} \\
&\quad - \frac{K}{2} \delta^{ns} (\frac{\partial}{\partial x^m} g_{is} + \frac{\partial}{\partial x^i} g_{ms} - \frac{\partial}{\partial x^s} g_{mi})(\delta_{nk}\delta_{jl} - \delta_{nl}\delta_{jk}) \\
&\quad - \frac{K}{2} \delta^{ns} (\frac{\partial}{\partial x^m} g_{js} + \frac{\partial}{\partial x^j} g_{ms} - \frac{\partial}{\partial x^s} g_{mj})(\delta_{ik}\delta_{nl} - \delta_{il}\delta_{nk}) \\
&\quad - \frac{K}{2} \delta^{ns} (\frac{\partial}{\partial x^m} g_{ks} + \frac{\partial}{\partial x^k} g_{ms} - \frac{\partial}{\partial x^s} g_{mk})(\delta_{in}\delta_{jl} - \delta_{il}\delta_{jn}) \\
&\quad - \frac{K}{2} \delta^{ns} (\frac{\partial}{\partial x^m} g_{ls} + \frac{\partial}{\partial x^l} g_{ms} - \frac{\partial}{\partial x^s} g_{ml})(\delta_{ik}\delta_{jn} - \delta_{in}\delta_{jk}) \\
&= 0. \qquad (73)
\end{aligned}$$

□

Thus hyperbolic space, which has constant strictly negative curvature, is a suitable candidate for N_1.

Remark 6.3. In fact hyperbolic space is the only simply connected Riemannian manifold M with constant negative curvature. It also is essentially the only complete one, apart from quotient spaces with respect to the action of certain subgroups of the group of isometries of M, see Petersen [13].

There are several ways to represent hyperbolic space. A common one is the upper half space representation

$$\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n > 0\}$$

with the metric

$$g_{ij}(x_1, \ldots, x_n) = \frac{\delta_{ij}}{x_n^2}.$$

From now on we will restrict ourselves to the dimension $n = 3$.

However, the way by which we will describe hyperbolic space is using *cylindrical coordinates.* These coordinates are constructed as follows: Start at some point x_0 in $\mathbb{H}^3$, choose some direction, and let $\gamma(t)$ be the geodesic in that direction. Also choose an orthonormal basis e_1, e_2 for the subspace orthogonal to γ at x_0, and parallel transport this along γ to get an orthonormal basis $e_1(t), e_2(t)$ for the subspace orthogonal to $\dot{\gamma}(t)$ for each t. At $\gamma(t)$ choose an angle θ and a radial distance r and walk the distance r along the geodesic starting at $\gamma(t)$ in direction $\cos(\theta)e_1(t) + \sin(\theta)e_2(t)$.

Consequently, cylindrical coordinates parametrize $\mathbb{H}^3$ according to the mapping

$$\Phi(r, \theta, t) = \exp_{\gamma(t)}(r\cos(\theta)e_1(t) + r\sin(\theta)e_2(t)).$$

Remark 6.4. Observe that $\exp_{\gamma(t)}$ is defined for all $r \in \mathbb{R}^+$ since $\mathbb{H}^3$ is geodesically complete and the injectivity radius is infinite. For more details on hyperbolic space see [2].

The metric takes a nice form in cylindrical coordinates, which is is well-known and stated without proof.

Lemma 6.5. *In cylindrical coordinates given as above by* $(r, \theta, t) \mapsto \Phi(r, \theta, t) \in \mathbb{H}^3$ *the metric of* $\mathbb{H}^3$ *takes the form*

$$g_{11} = \cosh^2(r), \; g_{22} = 1, \; g_{33} = \sinh^2(r), \; g_{12} = g_{13} = g_{23} = 0.$$

with respect to the natural frame $\xi_1 := \frac{\partial}{\partial t}, \quad \xi_2 := \frac{\partial}{\partial r}, \quad \xi_3 := \frac{\partial}{\partial \theta}$.

We determine the geometry of $\mathbb{H}^3$ in terms of these coordinates, starting with the computation of the Levi-Civita connection via the Christoffel symbols. Since we consider $\mathbb{H}^3$ as ambient space, we denote all geometric quantities by a bar according to our conventions in the previous chapters, that is

$$\bar{\Gamma}_{ij}^k = \frac{1}{2}\bar{g}^{km}\left(\frac{\partial}{\partial x^i}\bar{g}_{mj} + \frac{\partial}{\partial x^j}\bar{g}_{im} - \frac{\partial}{\partial x^m}\bar{g}_{ij}\right).$$

Lemma 6.6. *In terms of the above frame* $\{\xi_i\}$ *the Christoffel symbols of* $\mathbb{H}^3$ *all vanish apart from*

$$\begin{aligned}
\bar{\Gamma}_{12}^1 &= \bar{\Gamma}_{21}^1 = \frac{\sinh(r)}{\cosh(r)}, \quad \bar{\Gamma}_{11}^2 = -\cosh(r)\sinh(r) \\
\bar{\Gamma}_{33}^2 &= -\cosh(r)\sinh(r), \quad \bar{\Gamma}_{23}^3 = \bar{\Gamma}_{32}^3 = \frac{\cosh(r)}{\sinh(r)}.
\end{aligned}$$

Proof. Straightforward calculation. □

Furthermore we have

Lemma 6.7. *In terms of the above frame* $\{\xi_i\}$ *the Riemann tensor of* $\mathbb{H}^3$ *has the components*

$$\bar{R}_{1212} = -\cosh^2(r), \quad \bar{R}_{1313} = -\cosh^2(r)\sinh^2(r), \quad \bar{R}_{2323} = -\sinh^2(r). \tag{74}$$

Again all other components vanish.

Proof. This is also a straightforward calculation based upon formula (5). □

Remark 6.8. According to Lemma 6.2 the Riemann tensor is indeed parallel, i.e. $\nabla R \equiv 0$, since we have constant sectional curvature due to

$$K = \frac{\bar{R}_{1212}}{g_{11}g_{22}} = \frac{\bar{R}_{1313}}{g_{11}g_{33}} = \frac{\bar{R}_{2323}}{g_{22}g_{33}} = -1.$$

Having the geometry of $\mathbb{H}^3$ at our disposal, we now construct a convex hypersurface losing convexity.

6.2 Constructing a hypersurface

In this section we first compute the second fundamental form of "rotational" hypersurfaces. Afterwards we construct a suitable radial function, which determines a smooth compact surface without boundary losing convexity under mean curvature flow.

For the construction we may take advantage of the structure of the cylindrical coordinates by looking at rotational surfaces M of the shape

$$M = \{F(\varphi, s) \mid s \in \mathbb{R},\ \varphi \in [-\pi, \pi[\} \subset \mathbb{H}^3, \quad \text{with } F(\varphi, s) := \Phi(r(s), \varphi, s), \tag{75}$$

where the radius r depends smoothly on s. Observe that since $\mathbb{H}^3$ is geodesically complete, we may let s run all over the real axis, whereas φ is an angular coordinate chosen from $[-\pi, \pi[$.

Let $p \in M$ with $p = F(\varphi, s)$. Suppose that $r(s)$ is chosen in a way such that F is a smooth immersion $[-\pi, \pi[\times I \to \mathbb{H}^3$ for some interval $I \subset \mathbb{R}$. Then we have a natural frame of M given by the vector fields

$$e_1 := \frac{\partial F}{\partial \varphi} = DF\left(\frac{\partial}{\partial \varphi}\right) \quad \text{and} \quad e_2 := \frac{\partial F}{\partial s} = DF\left(\frac{\partial}{\partial s}\right),$$

which provides a basis $\{e_1(p), e_2(p)\}$ to work with at T_pM.

As usual we endow M with the metric g induced by $\mathbb{H}^3$, thus turning F into an isometry. Consequently, we define $g(e_i, e_j) := \bar{g}(e_i, e_j)$ for $i, j \in \{1, 2\}$.

We relate the geometry of M to the ambient geometry of $\mathbb{H}^3$ by means of the second fundamental form.

For this purpose observe that the cylindrical coordinates (r, θ, t) provide the natural frame

$$\xi_1 := \frac{\partial}{\partial t}\Phi, \quad \xi_2 := \frac{\partial}{\partial r}\Phi, \quad \xi_3 := \frac{\partial}{\partial \theta}\Phi.$$

of $\mathbb{H}^3$. For each $p \in M \subset \mathbb{H}^3$ we thus can express the vectors $e_i(p)$, $i = 1, 2$, in terms of the basis $\{\xi_j\}$. This yields

$$e_1(p) = \xi_3(p) \quad \text{and} \quad e_2(p) = \xi_1(p) + r'(s)\xi_2(p).$$

How do the vectors e_i change *within the tangent bundle of* $\mathbb{H}^3$, that is, seen from the viewpoint of the ambient space?

This information is provided by the covariant derivations of local extensions of e_1, e_2, say $\tilde{e}_1, \tilde{e}_2$. Recall the fact that covariant derivatives are pointwise objects which do not depend on the particular choice of the extensions. We compute at $p = F(\varphi, s) \in M$:

$$\begin{aligned}
\bar{\nabla}_{\tilde{e}_2}\tilde{e}_2 &= \bar{\nabla}_{\tilde{e}_2}(\xi_1 + r'(s)\xi_2) \\
&= \bar{\nabla}_{\tilde{e}_2}\xi_1 + r''(s)\xi_2 + r'(s)\bar{\nabla}_{\tilde{e}_2}\xi_2 \\
&= \bar{\nabla}_{\xi_1}\xi_1 + r'(s)\bar{\nabla}_{\xi_2}\xi_1 + r''(s)\xi_2 + r'(s)(\bar{\nabla}_{\xi_1}\xi_2 + r'(s)\bar{\nabla}_{\xi_2}\xi_2) \\
&= \bar{\Gamma}^k_{11}\xi_k + r'(s)\bar{\Gamma}^k_{21}\xi_k + r''(s)\xi_2 + r'(s)\bar{\Gamma}^k_{12}\xi_k + (r'(s))^2\bar{\Gamma}^k_{22}\xi_k \\
&= 2r'(s)\frac{\sinh(r(s))}{\cosh(r(s))}\xi_1 + [r''(s) - \cosh(r(s))\sinh(r(s))]\,\xi_2.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\bar{\nabla}_{\tilde{e}_1}\tilde{e}_1 &= \bar{\nabla}_{\xi_3}\xi_3 = \bar{\Gamma}^k_{33}\xi_k = -\cosh(r(s))\sinh(r(s))\xi_2 \\
\bar{\nabla}_{\tilde{e}_2}\tilde{e}_1 &= \bar{\nabla}_{\xi_1}\xi_3 + r'(s)\bar{\nabla}_{\xi_2}\xi_3 = (\bar{\Gamma}^k_{13} + r'(s)\bar{\Gamma}^k_{23})\xi_k = r'(s)\frac{\cosh(r(s))}{\sinh(r(s)}\xi_3.
\end{aligned}
$$

Finally we need a unit normal at p.

If we take into account that the metric g_{ij} is orthogonal, it is easy to verify that the normal space $\operatorname{span}(e_1(p), e_2(p))^\perp \subset T_p\mathbb{H}^3$ is given by $\mathbb{R}\tilde{\nu}(p)$ with

$$
\tilde{\nu}(p) := -r'(s)\xi_1(p) + \cosh^2(r(s))\xi_2(p).
$$

In particular, $\nu := \left(\sqrt{g(\tilde{\nu},\tilde{\nu})}\right)^{-1}\tilde{\nu}$ is a unit normal.

This yields

Lemma 6.9. *Let M be a rotational surface as in* (75)*. With the above choice of unit normal the components of the second fundamental form h_{ij} of M at $p \in M$ are given by*

$$
\begin{aligned}
h_{22} &= \frac{-r''(s)\cosh(r(s)) + 2(r'(s))^2\sinh(r(s)) + \cosh^2(r(s))\sinh(r(s))}{\sqrt{1+(r'(s))^2}} \\
h_{11} &= \frac{\cosh^2(r(s))\sinh(r(s))}{\sqrt{1+(r'(s))^2}} \\
h_{12} &= h_{21} = 0.
\end{aligned}
$$

Proof. This is an easy calculation using the Weingarten equations

$$
h_{ij} = -\bar{g}(\nu, \bar{\nabla}_{\tilde{e}_i}\tilde{e}_j).
$$

Observe that $\sqrt{g(\tilde{\nu},\tilde{\nu})} = \cosh(r(s))\sqrt{1+(r'(s))^2}$. □

Before we construct a suitable radial function r, we provide the following useful

Lemma 6.10. *Let $p = F(\varphi, 0) \in M$. Furthermore let $r'(0) = 0$. Then the Christoffel symbols of the induced Levi-Civita connection of M vanish at p.*

Proof. It is a basic fact from differential geometry that the induced connection ∇ of M can be interpreted as the orthogonal projection of covariant derivatives of tangent vectors of M onto the tangential bundle TM of M, where the covariant derivative is taken with respect to the ambient space.

Working in the frame e_1, e_2 as above, we have the basis $\{e_1(p), e_2(p)\}$ of T_pM. Since $T_pM \subset T_p\mathbb{H}^3$ we may decompose $T_p\mathbb{H}^3$ into the direct sum of T_pM and its orthogonal complement $(T_pM)^\perp$ by means of the inner product $g(p)$. Let $\pi^\top(v)$ be the projection of vectors $v \in T_p\mathbb{H}^3$ onto T_pM along $(T_pM)^\perp$. Then ∇ is determined by

$$
\nabla_{e_i}e_j := \pi^\top(\bar{\nabla}_{\tilde{e}_i}\tilde{e}_j), \quad i,j \in \{1,2\},
$$

where $\tilde{e}_i, \tilde{e}_j$ denote arbitrary extensions of e_i, e_j.

In our particular case it immediately follows that for $r'(0) = 0$ we have at p

$$
\bar{\nabla}_{\tilde{e}_i}\tilde{e}_j \in \operatorname{span}\{\xi_2(p)\} \quad \text{for } i,j \in \{1,2\}.
$$

Moreover, for $r'(0) = 0$

$$e_1(p), e_2(p) \in \operatorname{span}\{\xi_1(p), \xi_3(p)\}.$$

Since the $\{\xi_i\}$ are orthogonal and $T_pM = \operatorname{span}\{e_1(p), e_2(p)\}$ it therefore follows at p that

$$\nabla_{e_i} e_j = 0 \quad \text{for } i, j \in \{1, 2\}.$$

Thus the Christoffel symbols vanish at p. □

Now we are ready to construct the radial function r.

Theorem 6.11. *There is an interval $I := [-x_0, x_0]$, $x_0 > 0$, and a smooth function $r : I \to \mathbb{R}$ such that the following conditions are fulfilled:*

i) The mapping $F : [-\pi, \pi[\times\mathbb{R} \to \mathbb{H}^3, \quad F(\varphi, s) := \Phi(r(s), \varphi, s)$ smoothly parametrizes a convex, closed hypersurface M.

ii) At the points $F(\varphi, 0)$, $\varphi \in [-\pi, \pi[$ the second fundamental form h_{ij} of M takes a negative eigenvalue as soon as M is moved by mean curvature flow.

Remark 6.12. The main idea the following proof is based upon is to choose the first four derivatives of r at $s = 0$ in such a way that h_{ij} has a null-eigenvector at the points $F(\varphi, 0)$ and such that the evolution equation for h_{ij} (lemma 3.7) gets negative. From this we conclude that M must take a strictly negative eigenvalue at these points as soon as it is moved by mean curvature flow. The choice of these derivatives, however, should not impact the convexity of M around $F(\varphi, 0)$. Therefore the crucial negative portion in the evolution equation has to be contributed by the ambient curvature terms, which is possible because of the negative sectional curvature of $\mathbb{H}^3$.

Proof. The proof consists of two parts: First we derive the conditions on r to guarantee the desired behaviour at $s = 0$, and secondly we will show that we have the additional possibilitiy of choosing r in a way such that M is smooth, closed and convex everywhere else.

We suppose $r(0) = 1$ and $r'(0) = 0$. Furthermore let $p = F(\varphi, 0)$ for any φ.

Observe that lemma 6.9 yields $h_{12} = h_{21} = 0$, as well as $h_{11} > 0$. Due to $r(0) = 1$ and $r'(0) = 0$ it also shows that we have $h_{22} = 0$ at p, provided that

$$-r''(0)c_1 + c_1^2 c_2 = 0, \quad \text{with } c_1 = \cosh(1) \text{ and } c_2 = \sinh(1)$$

at p. This is equivalent to the condition

$$r''(0) = c_1 c_2, \tag{76}$$

which we assume to be true from now on. Note that M then is convex, but not strictly convex at p.

Under the above assumptions we want M to lose its convexity at p under mean curvature flow. For this purpose we show, provided that a solution exists, that h_{ij} will take a strictly negative principal curvature at p. Here we invoke the Hurwitz criterion, which states that h_{ij} is positive semi-definite if and only if both h_{11} and $\det h_{ij} = h_{11}h_{22} - h_{12}^2$ are greater than or equal to zero. Since $h_{11} > 0$ and $\det h_{ij} = 0$ at p, it is sufficient to show that

$$\frac{\partial}{\partial t}(\det h_{ij}) = h_{11}\frac{\partial}{\partial t}h_{22} + h_{22}\frac{\partial}{\partial t}h_{11} - 2h_{12}\frac{\partial}{\partial t}h_{12} = h_{11}\frac{\partial}{\partial t}h_{22} < 0 \quad \text{at } p.$$

Since $h_{11} > 0$ at p, it remains to show that $\frac{\partial}{\partial t} h_{22} < 0$ at p. To this end we take into account that $\mathbb{H}^3$ is locally symmetric ($\bar{\nabla}\bar{R} \equiv 0$) and that $h_{22} = h_{12} = h_{21} = 0$. Then the evolution equation for h_{22} (theorem 3.7) takes at p the simple form

$$\frac{\partial}{\partial t} h_{22} = \Delta h_{22} + 2h_{lk} g^{l1} g^{k1} \bar{R}(e_1, e_2, e_1, e_2), \tag{77}$$

which we want to take a negative value.

Let us consider Δh_{22}. Observe that because of $r'(0) = 0$ lemma 6.10 applies and all Christoffel symbols of the induced connection of M vanish at p. Applying this to the general formula for covariant derivatives of tensors, the second derivatives of h_{ij} simplify to

$$\nabla_{e_l} \nabla_{e_k} h_{ij} = e_l(e_k(h_{ij})) - e_k(\Gamma^n_{li} h_{nj}) - e_k(\Gamma^n_{lj} h_{in}) - e_l(\Gamma^n_{ki} h_{nj}) - e_l(\Gamma^n_{kj} h_{in}) \quad \text{at } p.$$

Applying the product rule and the assumption that $h_{22} = 0$ we get

$$\nabla_{e_l} \nabla_{e_k} h_{22} = e_l(e_k(h_{22})) \quad \text{and} \quad \Delta h_{22} = g^{lk} e_l(e_k(h_{22})) \quad \text{at } p.$$

Recall now that $e_1 = DF(\frac{\partial}{\partial \varphi})$ and $e_2 = DF(\frac{\partial}{\partial s})$. Since h_{22} depends on s but not on φ, all second derivatives of h_{22} vanish except for $e_2(e_2(h_{22}))$, which is simply the second derivative of h_{22} with respect to s at p. If we differentiate the expression of h_{22} given in lemma 6.9, we obtain with $r(0) = 1$ and $r'(0) = 0$

$$\begin{aligned} e_2(e_2(h_{22})) &= -c_1 r''''(0) + (r''(0))^3 c_1 + (3c_2 - c_1^2 c_2)(r''(0))^2 + (3c_1^3 - 2c_1) r''(0) \\ &= -c_1 r''''(0) + 3c_1^2 c_2^3 + 3c_1^4 c_2 - 2c_1^2 c_2, \end{aligned}$$

where in last identity we have employed $r''(0) = c_1 c_2$.

Note that we have $e_1(p) = \xi_3(p)$ and $e_2(p) = \xi_1(p)$. Thus $g^{11} = \bar{g}^{33} = (\sinh^2(1))^{-1}$ and $g^{22} = \bar{g}^{11} = (\cosh^2(1))^{-1}$ at p. All other components of g^{ij} vanish. Moreover, from lemma 6.9 and $r(0) = 1$, $r'(0) = 0$ it follows that $h_{11} = c_1^2 c_2$. Finally, $\bar{R}(e_1, e_2, e_1, e_2)$ $= \bar{R}_{3131} = -c_1^2 c_2^2$ by (74). Thus (77) takes the form

$$\begin{aligned} \frac{\partial}{\partial t} h_{22} &= \frac{1}{c_1^2} e_2(e_2(h_{22})) + 2\frac{h_{11}}{c_2^4} \bar{R}_{3131} \\ &= \frac{1}{c_1^2}[-c_1 r''''(0) + 3c_1^2 c_2^3 + 3c_1^4 c_2 - 2c_1^2 c_2] - 2\frac{c_1^4}{c_2} \quad \text{at } p. \end{aligned} \tag{78}$$

Clearly the latter term $2h_{11}\bar{R}_{3131}/c_2^4$ is strictly negative. It is therefore possible to choose $r''''(0)$ such that $\Delta h_{22} = e_2(e_2(h_{22})) = \vartheta > 0$ and $\frac{\partial}{\partial t} h_{22} < 0$ at p. This is for instance the case for

$$r''''(0) = 2. \tag{79}$$

Thus we have derived the sufficient conditions ensuring that M is convex at p and h_{ij} takes a negative eigenvalue as soon as M is moved by mean curvature flow.

Observe that all assumptions we have made on r up to now are satisfied by the symmetric, smooth function

$$r(s) := \sqrt{1 - a_1 s^2 - a_2 s^4 - a_3 s^6}$$

for appropriate choices of the coefficients a_i. Obviously, $r(0) = 1$ and $r'(0) = 0$. Moreover, since $r''(0) = -a_1$ we choose $a_1 = -c_1 c_2$ in order to satisfy (76). In addition to that we want to satisfy (79). Since $r''''(0) = -3a_1^2 - 12a_2$ we accordingly choose $a_2 = (-3c_1^2c_2^2 - 2)/12$.

Furthermore note that if we choose $a_3 > 0$ there are points $\pm x_0 \in \mathbb{R}$ with $r(-x_0) = r(x_0) = 0$ and $r(s) > 0$ for all $-x_0 < s < x_0$. We set $I := [-x_0, x_0]$. Then M is compact and boundary-free.

It remains to check that M is smooth and convex for this choice of r. As to smoothness it is sufficient to consider the points $F(-x_0, \varphi)$ and $F(x_0, \varphi)$, respectively. Note that due to the squareroot, $r'(s)$ tends to $\pm\infty$ as we let $s \to \mp x_0$. Thus the mapping F indeed parametrizes a hypersurface M which is smooth everywhere.

In order to prove that M is convex away from the points $F(0, \varphi)$ we first show that M is even strictly convex on $]-x_0, x_0[\setminus\{0\}$. Due to lemma 6.9 this is equivalent to $h_{22} > 0$ for $i = 1, 2$, since h_{11} is strictly positive everywhere.

We have $r(-s) = r(s)$ for all $s \in I$. Then, for reasons of symmetry, the function $h_{22} : s \mapsto h_{22}(s)$ is symmetric, too, that is $h_{22}(-s) = h_{22}(s)$ on I. Since r is smooth, h_{22} has a horizontal tangent at $s = 0$, and thus a local minimum because of $e_2(e_2(h_{22})) = \vartheta > 0$ at p. Moreover, it is easy to verify that for our choice of r this is also a global minimum on I. Thus we have $h_{22} > 0$ on $]-x_0, x_0[\setminus\{0\}$.

Finally it also follows that M is convex at the points $F(-x_0, \varphi)$ and $F(x_0, \varphi)$ due to continuity reasons. $\square$

7 Example II: Berger spheres

In this chapter we turn our interest to the second situation we have announced: In an appropriate ambient space N_2 with nonnegative sectional curvature and which is *not locally symmetric*, i.e. $\nabla \bar{R} \not\equiv 0$, we construct a convex hypersurface immediately losing its convexity when moved by mean curvature flow. However, in this case we only give a *local* description of the surface, whence we end up with a slightly different conclusion as in the hyperbolic space example.

In contrast to the previous chapter, a great deal of work is devoted to the introduction and discussion of the three-dimensional ambient spaces we work with: the *Berger spheres*. The present chapter therefore is mainly divided into two parts.

In the first part we solely concentrate on the ambient space. Basically, Berger spheres are distorted 3-spheres endowed with a 1-parameter family of *Berger metrics* whose definition is closely related to the so-called *Hopf fibration* of the 3-sphere S^3. We therefore first describe the Hopf map and the Hopf fibration. Then we define and closely examine the Berger spheres, revealing their Lie group nature and relating them to S^3. Furthermore we provide a few important facts from basic Lie group theory. Finally we compute the important differential geometric quantities of Berger spheres, such as the Levi-Civita-connection and the curvature.

In the second part we start by examining the geometry of a natural submanifold of codimension 1. Then, using a graph approach over this submanifold, we locally construct a convex hypersurface losing convexity under mean curvature flow. Again, convexity is meant in the sense of definition 4.1.

7.1 Hypersphere and Hopf map

Our goal in this section is to provide both topological and geometrical information about the hypersphere or 3-sphere. We motivate the definition of the Hopf map and show that it is an isometric submersion.

Definition 7.1. The 3-*sphere* S^3 is the set of all points in four-dimensional Euclidean space with unit distance from the origin given by

$$S^3 := \{(a, b, c, d) \in \mathbb{R}^4 \mid a^2 + b^2 + c^2 + d^2 = 1\}.$$

Identifying $\mathbb{R}^4 \simeq \mathbb{C}^2$, we may also write

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

Let us describe important subsets of S^3.

Definition 7.2. A *(k-)subsphere* of S^3 is the intersection of S^3 with a $(k+1)$-dimensional vector subspace of $\mathbb{R}^4$ for $k = 1, 2$. A 1-subsphere is also called a *great circle* and a 2-subsphere is called *equator*.

A fruitful way to understand higher dimensional objects like S^3 is to use projections or cross-sections of these objects to obtain lower dimensional pictures. In the following we consider a special projection of S^3, called the *Hopf map*. The Hopf map induces a fibration of S^3, that is a decomposition in non-intersecting great circles. This *Hopf fibration*, in turn, gives rise to the definition of the Berger spheres we introduce later on.

As the Hopf map is an interesting topic for itself, we will elaborate on it somewhat further in this section. In particular we show that it is an isometric (or Riemannian) submersion (see Petersen [13] for further references).

We start by considering mappings from $\mathbb{R}$ to S^3 given by

$$t \mapsto (e^{it}z_1, e^{it}z_2),$$

for fixed $(z_1, z_2) \in S^3$.

If we think of $\mathbb{C}^2$ as a $\mathbb{C}$-linear vector space, this is simply inner multiplication by elements of S^1.

Observe that these mappings define an *action* of S^1 on S^3 by assigning to each $e^{it} \in \mathbb{R}$ the mapping φ_t from S^3 onto itself defined by

$$\varphi_t : (z_1, z_2) \mapsto (e^{it}z_1, e^{it}z_2), \quad (z_1, z_2) \in S^3, \quad \text{with} \quad \varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}.$$

Definition 7.3. The *orbits* of this action, that is the sets $\{(e^{it}z_1, e^{it}z_2) | e^{it} \in S^1\}$, are called *Hopf fibers.*

Lemma 7.4. *Hopf fibers are great circles and for every $z = (z_1, z_2) \in S^3$ there is a unique Hopf fiber running through z.*

Proof. Let $z = (z_1, z_2) \in S^3$. The Hopf fiber $\{(e^{it}z_1, e^{it}z_2),\ t \in \mathbb{R}\}$ lies in the one-dimensional $\mathbb{C}$-linear subspace V of $\mathbb{C}^2$ spanned by z. Indeed it is just the intersection of V with S^3. V is uniquely determined and can be considered as a two-dimensional $\mathbb{R}$-linear vector subspace of $\mathbb{R}^4$, which proves the lemma according to definition 7.2. □

We thus can think of S^3 as a set (or *fibration*) of non-intersecting great circles, the Hopf fibers. It is easy to see that by this structure we have given an equivalence relation $\sim_R$ on S^3 by identifying all points lying in the same Hopf fiber. Precisely:

$$z \sim_R z' \text{ if and only if } z = e^{it}z' \text{ for some } t \in \mathbb{R}.$$

A natural idea is now to try to somehow project S^3 along its Hopf fibers onto some two-dimensional object.

For this purpose let us consider the one-dimensional complex projective space $P^1(\mathbb{C})$. This is the set of all equivalence classes of the equivalence relation on $\mathbb{C}^2 \backslash \{0\}$ given by $z \sim z'$ if $z = \lambda z'$ for some $\lambda \in \mathbb{C}$. Restricted on $S^3 \subset \mathbb{C}^2$, this equivalence relation is just the same as $\sim_R$ defined above.

Definition 7.5. The *natural projection* $\pi_\sim$ with respect to the above equivalence relation $\sim$ is a mapping

$$\pi_\sim : \mathbb{C}^2 \to P^1(\mathbb{C})$$

which maps each $z \in \mathbb{C}^2$ to the point $\pi_\sim(z) := \mathbb{C}z \in P^1(\mathbb{C})$. It is called *Hopf map* when restricted to S^3.

Remark 7.6. The Hopf fibration is the analogon to the one-dimensional fibration of the 1-sphere $S^1 \subset \mathbb{R}^2$, which is induced by the restriction of the natural projection $\phi : \mathbb{R}^2 \to P^1(\mathbb{R})$ with $\phi(x) := \mathbb{R}x$ on S^1. Here the fibers are just antipodal points $\{p, -p\}$ for $p \in S^1$.

Remark 7.7. $P^1(\mathbb{C})$ is topologically identical with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Moreover, the stereographic projection provides us with a tool to diffeomorphically identify S^2 with $\hat{\mathbb{C}}$. The Hopf map thus can be thought of a diffeomorphism from S^3 to S^2.

The somewhat abstract definition of the Hopf map $\pi_\sim : S^3 \to P^1(\mathbb{C}) \simeq S^2$ given above can be made clear by choosing suitable coordinates both for S^3 and S^2. An appropriate choice of frame additionally shows that it is a Riemannian submersion, as we demonstrate in the sequel.

We may represent a point $p \in S^3$ by $p = (e^{i\theta_1} \sin t, e^{i\theta_2} \cos t)$. This representation describes the position of $p \in S^3 \subset \mathbb{C}^2$ by specifying its position in two affine subspaces parallel to $\{c = (c_1, c_2) | c_1 = 0\}$ and $\{c = (c_1, c_2) | c_2 = 0\}$, respectively. The sections of S^3 with these subspaces are 1-spheres. This suggests a parametrization of S^3 by

$$\Phi : (t, e^{i\theta_1}, e^{i\theta_2}) \mapsto (e^{i\theta_1} \sin t, e^{i\theta_2} \cos t),$$

which describes the imbedding

$$\Phi :]0, \pi/2[\times S^1 \times S^1 \hookrightarrow S^3(1) \subset \mathbb{C}^2.$$

By this parametrization the 3-sphere is seen as some sort of twofold rotational surface.

The imbedding Φ can be made isometric if we write the metric on S^3 as

$$g_1 = dt^2 + \sin^2 t d\theta_1^2 + \cos^2 t d\theta_2^2,$$

where $d\theta_i$ are the standard metrics on the two 1-spheres. Obviously this metric, which is a so-called *doubly warped product*, coincides with the standard metric on S^3 induced by $\mathbb{R}^4$.

Remark 7.8. The reason for the choice of these coordinates is the fact that on $]0, \pi/2[\times S^1 \times S^1$ the Hopf fibration takes the clear form

$$\theta \mapsto (t, e^{i(\theta_1 + \theta)}, e^{i(\theta_2 + \theta)}).$$

Our intention is to describe the Hopf map as a mapping $S^3 \to S^2$. We may even show that it is a Riemannian (or isometric) submersion if we construct it as a mapping $S^3 \to S^2(\frac{1}{2})$. Recall that an isometric submersion is a submersion whose differential at each point p restricted to its cokernel is a linear isometry.

We now consider $S^2(\frac{1}{2})$. The trick is to represent $S^2(\frac{1}{2})$ in an analogous way as we represented S^3 above. A similar reasoning allows us to parametrize $S^2(\frac{1}{2})$ by

$$\Psi : (r, e^{i\theta}) \to \left(\frac{1}{2} \cos(2r), \frac{1}{2} e^{i\theta} \sin(2r) \right), \quad r \in [0, \pi/2],$$

where the somewhat strange choice of coefficients will be explained later. Observe that in these coordinates the induced metric takes the form

$$g_2 = dr^2 + \frac{\sin^2(2r)}{4} d\theta^2.$$

Now recall that we have defined the Hopf map as the quotient mapping of the equivalence relation induced by identifying points on the same Hopf fiber. Notice that in our parametrization Φ of S^3 two points $p_1 = (t_1, e^{i\theta_1}, e^{i\theta_2})$, $p_2 = (t_2, e^{i\vartheta_1}, e^{i\vartheta_2})$ lie on the same Hopf fiber if and only if $\theta_1 - \theta_2 = \vartheta_1 - \vartheta_2$ and $t_1 = t_2$.

Hence the Hopf map can be written as the mapping $S^3 \to S^2(1/2)$ with $(t, e^{i\theta_1}, e^{i\theta_2}) \to (t, e^{i(\theta_1 - \theta_2)})$.

In addition to that it is possible to find frames for S^3 and $S^2(1/2)$ by means of which it gets clear that the Hopf map is an isometric submersion. For this purpose we first note that the vector field $v_0 := \partial_{\theta_1} + \partial_{\theta_2}$ is tangent to the Hopf fibers on S^3. Evidently, the vector field $v_1 := \partial_t$ is orthogonal to it and has unit length. Next observe that we can easily construct another vector field v_2 of unit length perpendicular to v_0 and v_1 namely

$$v_2 = \frac{\cos^2 t \partial_{\theta_1} - \sin^2 t \partial_{\theta_2}}{\cos t \sin t}.$$

By this construction we see that v_1 and v_2 span the cokernel of the Hopf map.

On the other hand we have an orthonormal frame on $S^2(1/2)$, namely

$$w_1 = \partial_r \quad \text{and} \quad w_2 = \frac{2}{\sin(2r)} \partial_\theta.$$

It is clear that the Hopf map takes v_1 to w_1 and

$$v_2 = \frac{\cos^2 t \partial_{\theta_1} - \sin^2 t \partial_{\theta_2}}{\cos t \sin t} \to \frac{\cos^2 r \partial_\theta + \sin^2 r \partial_\theta}{\cos r \sin r} = \frac{2}{\sin(2r)} \partial_\theta = w_2,$$

which shows that it indeed is an isometric submersion.

Remark 7.9. The above considerations reveal that the geometry perpendicular to the kernel of the differential of the Hopf map does not get affected. Only the direction along the Hopf fibers "collapses". This idea contributes to the definition of Berger spheres, as we will see.

The next section points out another very important way to represent S^3.

7.2 The hypersphere as a Lie group

In this section we characterise the hypersphere as a Lie group. For this purpose we give a brief repetition of important notions from Lie group theory, such as left invariant vector fields and integral curves.

7.2.1 Lie Groups

Recall that a *Lie group* G is a topological group with a differentiable structure such that the group operations multiplication $* : G \times G \to G$ and inversion $^{-1} : G \to G$ are differentiable mappings.

From the manifold viewpoint, the tangent space T_eG of G at the neutral element e plays an important role, since in addition to its vector space structure it also carries the structure of a *Lie algebra*. We recall:

Definition 7.10. An (abstract) Lie algebra is a vector space V endowed with a *Lie bracket operation* $[\cdot,\cdot] : V \times V \to V$, which has the properties

1. $[x, x] = 0$ and
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ *(Jacobi identity)*

Remark 7.11. In the (special) case of a Lie algebra T_eG belonging to a Lie group G, $[\cdot,\cdot]$ is given by the restriction of the bracket operation on tangent vector fields of G to T_eG.

7.2.2 Left invariant vector fields and metrics

There is a natural way to extend a metric g given for vectors in the Lie algebra T_eG of a Lie group G to the whole tangential bundle TG by taking advantage of the group structure of G. First we provide the important idea of considering the tangent vectors at e as left-invariant vector fields, that is, invariant under left translations.

Definition 7.12. Let G be a Lie group and $g \in G$. The mapping

$$\mathsf{L}_g : G \to G \text{ with } h \mapsto gh \text{ for all } h \in G$$

is called left translation by g.

Remark 7.13. By the fact that the group multiplication is a differentiable mapping, it follows that all left translations L_g are diffeomorphisms of G onto itself, whence the differential $D\mathsf{L}_g$ is well defined.

Definition 7.14. Let G be a Lie group with neutral element e. A vector field X is called *left-invariant* if the relation

$$X(g) = D\mathsf{L}_g(X(e))$$

holds for all $g \in G$.

This definition shows that we may identify each left-invariant vector field with its value at the neutral element $e \in G$. Similarly, left-invariant vector fields enable us to extend metrics given on T_eG: The *left-invariant* metric is the unique metric making left translations isometric, as we see in the following

Definition 7.15. A metric g of a Lie group G is called *left-invariant*, if we have

$$g(X(p), Y(p)) = g(X(e), Y(e))$$

for all left-invariant vector fields X, Y.

With all this in mind, we describe the hypersphere as a Lie group in the next section.

7.2.3 The special unitary group $SU(n)$

We focus on a special Lie group, the special unitary group $SU(n)$. Later it will serve as a starting point for the definition of Berger spheres.

Let $\mathbb{H}$ be the four-dimensional space of quaternions in matrix notation. $\mathbb{H}$ is spanned by the basis $\{E, I, J, K\}$ with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

which satisfy the relations $I^2 = J^2 = K^2 = -1,\ IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J$ and $KI = -IK = J$.

Remark 7.16. In contrast to the usual $\mathbb{H} \simeq \mathbb{R}^4$ point of view we also can consider $\mathbb{H}$ as the two-dimensional $\mathbb{C}$-linear space $\mathbb{H} = \mathbb{C} + J\mathbb{C}$ and consider its elements as 2×2-matrices with complex components of the form

$$\left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix},\ u, v \in \mathbb{C} \right\}.$$

If we define the norm of such a matrix as $\sqrt{|u|^2 + |v|^2}$, where $|\cdot|$ be the complex norm, it is easy to verify that the quaternions of norm 1 form a group with respect to matrix multiplication. We define

Definition 7.17. The *special unitary group* $SU(2)$ is defined by

$$\begin{aligned} SU(2) : &= \{A \in M_{2\times 2}(\mathbb{C}) | \det A = 1,\ A^{-1} = \bar{A}^t\} \\ &= \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} |\quad |u|^2 + |v|^2 = 1 \right\}. \end{aligned} \tag{80}$$

It is clear that as a set $SU(2)$ is just S^3. In the following we will therefore endow $SU(2)$ with the topology from S^3.

It is well-known that the special unitary group $SU(2)$ is a differentiable manifold. In combination with the matrix multiplication it therefore forms a Lie group, as stated in the following

Lemma 7.18. *SU(2) with the usual matrix multiplication forms a Lie group with neutral element $e = E$. The Lie algebra $T_eSU(2)$ is spanned by the matrices $X_1 = I$, $X_2 = J$, and $X_3 = K$.*

Proof. Obviously $SU(2)$ is a differentiable manifold parametrized by the matrix entries. The matrix multiplication and inversion are well-defined and can be described by polynomial equations involving the matrix entries and consequently are differentiable.

Thinking of $\mathbb{H}$ as a four-dimensional real vector space with respect to the (orthonormal) basis $\{E, I, J, K\}$, $SU(2) = S^3$ simply is the hypersphere tangent to the hyperplane $\text{span}(I, J, K)$ at E. Thus $T_eSU(2) = \text{span}(I, J, K)$. ☐

7.2.4 Integral curves and the exponential mapping

Before we further examine $SU(2)$ we need to know how left-invariant vector fields explicitly look like. Suppose we have given some Lie group G along with some vector $v \in T_eG$. Here and in the following we will denote the corresponding left-invariant vector field by $\tilde{v}$ and call it the *left-invariant extension* of v. How does $\tilde{v}(p)$ look like for an arbitrary point $p \in G$? In other words, which is the integral curve of $\tilde{v}$ running through p?

Definition 7.19. A curve $c(t)$ is called *integral curve* of the left-invariant vector field X, if

$$\frac{\partial}{\partial t} c(t) = X(c(t)) = D\mathsf{L}_{c(t)}(X(e)).$$

Once having found *one* integral curve $c(t)$ for $\tilde{v}$, it is easy to find others. In particular the following lemma shows how to find the one leading through a certain given point p.

Lemma 7.20. *Let G be a Lie group and $g \in G$. Furthermore let X be a left-invariant vector field. Suppose that $c(t)$ is an integral curve of X. Then $gc(t)$ is an integral curve of X as well.*

Proof. It is

$$\begin{aligned} \frac{\partial}{\partial t}(gc(t)) &= D\mathsf{L}_g(\frac{\partial}{\partial t}c(t)) = D\mathsf{L}_g(X(c(t))) = D\mathsf{L}_g(D\mathsf{L}_{c(t)}(X(e))) \\ &= D\mathsf{L}_{(gc(t))}(X(e)) = X(gc(t)). \end{aligned}$$

□

Consequently, $d(t) := (c(0))^{-1}c(t)$ is the curve leading through the neutral element e at $t = 0$, whereas $pd(t)$ is the one leading through p at $t = 0$.

Corollary 7.21. *Let X be a left-invariant vector field of $SU(2)$ and $A \in SU(2)$. Then $X(p) = pX(e)$.*

Proof. Let $c(t)$ be the integral curve with $c(0) = e$ and $\frac{\partial}{\partial t}c(t)|_{t=0} = X(e)$. Then

$$\begin{aligned} X(A) &= X(Ac(t))|_{t=0} = D\mathsf{L}_A(X(c(t)))|_{t=0} \\ &= \frac{\partial}{\partial t}(Ac(t))|_{t=0} = A\frac{\partial}{\partial t}(c(t))|_{t=0} = AX(e). \end{aligned}$$

□

We thus have a simple method to determine the values of a left-invariant vector field on $SU(2)$ by its value at e.

Example 7.22. A point $p = (\cos te^{i\theta_1}, \sin te^{i\theta_2}) \in S^3$ may be written as

$$p = a_1E + a_2I + b_1J + b_2K \in SU(2),$$

where $a_1 = \cos t \cos\theta_1$, $a_2 = \cos t \sin\theta_1$, $b_1 = \sin t \cos\theta_2$, and $b_2 = \sin t \sin\theta_2$. We want to determine the values of the left-invariant extensions of the basis vectors $X_1, X_2, X_3 \in T_eSU(2)$. Due to $X_1 = I$, $X_2 = J$, $X_3 = K$ and the relations $I^2 = J^2 = K^2 = -E$, $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$ and $KI = -IK = J$, we find by the above corollary

$$\begin{aligned} X_1(p) &= a_1I - a_2E - b_1K + b_2J \\ X_2(p) &= a_1J + a_2K - b_1E - b_2I \\ X_3(p) &= a_1K - a_2J + b_1I - b_2E. \end{aligned}$$

Later these relations will be particularly valuable in order to represent vectors in terms of $X_i(p)$.

As far as lemma 7.20 is concerned, the question still remains how to find *one* first integral curve, say, the one running through e. The answer to this question depends on the special structure of each Lie group under consideration - since in our case we are particularly interested in $S^3 \cong SU(2)$, we will focus on this Lie group now.

Recall from the theory of ordinary differential equations that for square matrices M with complex components the *exponential mapping* given by

$$\exp(tM) := \sum_{k=0}^{\infty} \frac{(tM)^k}{k!} \quad \text{for } t \in \mathbb{R}$$

is well-defined, that is, the series converges to a matrix. Moreover, it is

$$\frac{\partial}{\partial t} \exp(tM)|_{t=0} = M.$$

It is easy to verify that $\exp(tM) \in SU(2)$ for all $t \in \mathbb{R}$, if $M \in SU(2)$.

Accordingly, we have the following

Lemma 7.23. *An integral curve through e of the left-invariant vector field extension $\tilde{X}$ of a given vector $X \in T_eSU(2)$ is given by*

$$t \to \exp(tX) \in SU(2).$$

Remark 7.24. It is worth noting that the mapping $F_X : \mathbb{R} \to SU(2)$ defined by $F_X(t) := \exp(tX)$ is a 1-*parameter subgroup* of $SU(2)$, that is, it is both a diffeomorphism and a group homomorphism. The latter fact is due to the functional equation

$$\exp((t_1 + t_2)X) = \exp(t_1X) \exp(t_2X).$$

It can be shown that the 1-parameter subgroups of a Lie group are exactly the integral curves of left-invariant vector starting at the neutral element (see for example [11].)

Remark 7.25. The above considerations give rise to the definition of the *exponential mapping of a Lie group G* as a mapping taking vectors from the Lie algebra T_eG to G. This exponential map is closely related to the well-known exponential map for Riemannian manifolds - in fact for every Lie group there is a unique connection, the so-called *left-invariant connection*, making all integral curves of left-invariant vector fields geodesic, in which case the two exponential mappings are identical at e. Note, however, that the Berger-sphere connection introduced in the next section does not satisfy this condition.

In the case of $SU(2)$, the exponential mapping exp for Lie groups may be written down explicitly. We have

Lemma 7.26. *Let $v = v^iX_i \in T_eSU(2)$ with $(v^1)^2 + (v^2)^2 + (v^3)^2 = 1$. Let $t \geq 0$. Then the image of tv under the Lie group exponential map* $\exp : T_eSU(2) \to SU(2) \in \mathbb{H}$ *is given by*

$$\exp(tv) = \cos tE + \sin t(v^1I + v^2J + v^3K) \in SU(2).$$

Consequently, exp *is diffeomorphic for $t \in [0, \pi[$.*

Proof. It is

$$\exp(tv) = E + tv + \frac{t^2}{2}v^2 + \frac{t^3}{6}v^3 + \frac{t^4}{24}v^4 + \dots$$

where the powers of v are computed with respect to the matrix multiplication. Due to the relations $I^2 = J^2 = K^2 = -E$, $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$ and $KI = -IK = J$ the powers of v get periodic for increasing exponents, leading to the asserted expression. ☐

Example 7.27. As an application of this lemma we find that the integral curves of the left-invariant vector field X_1 are tangent to the Hopf fibers. For this purpose let us represent a point $p \in SU(2)$ by

$$p = \begin{pmatrix} \cos t e^{i\theta_1} & \sin t e^{-i\theta_2} \\ -\sin t e^{i\theta_2} & \cos t e^{-i\theta_1} \end{pmatrix}.$$

Furthermore, the lemma yields that the integral curve of $X_1 = I$ through E has the form

$$\theta \mapsto \cos\theta E + \sin\theta I = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\theta \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

To determine the integral curve of X_1 through the arbitrary point p we invoke lemma 7.20 and find that it takes the form

$$\theta \mapsto \begin{pmatrix} \cos t e^{i\theta_1} & \sin t e^{-i\theta_2} \\ -\sin t e^{i\theta_2} & \cos t e^{-i\theta_1} \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos t e^{i(\theta_1+\theta)} & \sin t e^{-i(\theta_2+\theta)} \\ -\sin t e^{i(\theta_2+\theta)} & \cos t e^{-i(\theta_1+\theta)} \end{pmatrix}.$$

Obviously this is exactly the Hopf fiber running through p (see definition 7.3). Hence the vector field X_1 is tangent to the Hopf fibers at any point $p \in SU(2)$. This fact contributes to the definition of Berger spheres, as we will see in the next section.

Remark 7.28. We can turn $SU(2)$ into a Riemannian manifold by assigning a left-invariant metric to it. We can do this by demanding that the left-invariant vector fields be an orthonormal frame. It is easy to verify that this choice of metric coincides with the induced metric from $\mathbb{R}^4$.

7.3 Berger spheres and their geometry

Now the way to understand the idea of Berger spheres is straightforward. We have seen from the Lie group viewpoint that at each point $p \in SU(2)$ a left-invariant metric can be defined with respect to the basis $X_i(p)$, $i = 1, 2, 3$, of $T_pSU(2)$, where X_i are the left-invariant vector fields corresponding to the basis vectors $X_i \in T_eSU(2)$, $i = 1, 2, 3$. Then, changing to the hypersphere viewpoint, it turned out that $X_1(p)$ is always tangent to the unique Hopf fiber running through p.

We also know that we have a Riemannian submersion, the Hopf map, which takes S^3 to S^2 by simply thinking of each Hopf fiber as a single point. So why not performing this process slowly by gradually shrinking the Hopf fibers from their original length down to a single point? This, of course, can be done by deliberately varying the standard metric of S^3 in such a way that all is kept the same except for the distances in direction of the Hopf fibers. As X_1 is tangent to the Hopf fibers and X_2, X_3 are perpendicular to it, we simply have to shrink X_1. This gives rise to the definition of the following 1-parameter family of *Berger metrics*.

Definition 7.29. On $SU(2)$ we define the 1-parameter family of Berger metrics g^ε at $p \in SU(2)$ by

$$g_{11}^\varepsilon := \varepsilon^2, \quad g_{22}^\varepsilon = g_{33}^\varepsilon = 1, \quad g_{ij}^\varepsilon = 0, \quad i \neq j,$$

with respect to the left-invariant basis $X_i(p)$, $p = 1, 2, 3$, defined as above, and $\varepsilon \in]0, 1]$. $SU(2)$ endowed with the metric g^ε is called Berger sphere S_ε^3.

Remark 7.30. By this definition we see that for $\varepsilon = 1$ we have the standard hypersphere as in remark 7.28, which gets more and more distorted for decreasing ε and will finally collapse as we let $\varepsilon \to 0$.

An interesting question is what happens to the geometry of S^3 if we decrease ε. What happens to the curvature and the geometry of submanifolds such as equators? For this purpose we examine the Riemannian connection and the curvature of S^3_ε.

The following calculations will be performed with respect to the orthogonal, left-invariant moving frame X_i, $i = 1, 2, 3$, which exists on all of $SU(2)$. However, this setting has the disadvantage that Lie-brackets do not vanish. In particular, since on $SU(2)$ Lie-brackets are computed by $[X, Y] = XY - YX$, it is at E

$$[X_i, X_{i+1}] = 2X_{i+2}, \tag{81}$$

where indices are mod 3 for reasons of simplicity. This can be readily verified using $X_1 = I$, $X_2 = J$ and $X_3 = K$.

Remark 7.31. According to the well-known Frobenius integrability theorem the non-vanishing Lie-brackets imply that there is no 2-submanifold M of S^3 such that $X_i(p)$, $X_j(p)$, $i \neq j$, is a basis for $T_pM \subset T_pS^3$ for all $p \in M$. This will be of importance later.

Consequently we do not have the situation as in local coordinates, and instead of computing the connection via the (symmetric) Christoffel symbols, we have to employ the Koszul formula which generalizes the definition of the Christoffel symbols:

Lemma 7.32. *The Levi-Civita connection ∇ of a Riemannian manifold (M, g) can be computed at $p \in M$ by*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &+ g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y), \end{aligned}$$

where X, Y, Z are (local) vector fields around p.

Proof. This is a direct consequence of the Levi-Civita connection being a derivation, tensorial, torsion-free, and making the metric a parallel tensor. □

By means of this formula we compute (using the bar notation from now on to indicate that S^3_ε will later be interpreted as an ambient space):

Lemma 7.33. *The Levi-Civita connection $\bar{\nabla}$ of S^3_ε is given by*

$$\begin{aligned} &\bar{\nabla}_{X_1} X_2 = (2 - \varepsilon^2) X_3, \quad \bar{\nabla}_{X_2} X_3 = X_1, \quad \bar{\nabla}_{X_3} X_1 = \varepsilon^2 X_2 \quad \textit{and} \\ &\bar{\nabla}_{X_2} X_1 = -\varepsilon^2 X_3, \quad \bar{\nabla}_{X_3} X_2 = -X_1, \quad \bar{\nabla}_{X_1} X_3 = (\varepsilon^2 - 2) X_2. \end{aligned}$$

Proof. Direct application of lemma 7.33 and definition 7.29. □

Furthermore, to compute the Riemannian curvature tensor, we cannot use the coordinate definition, but instead have to work with the coordinate-free version

$$\bar{R}(X_i, X_j)X_k := \bar{\nabla}_{X_i}\bar{\nabla}_{X_j}X_k - \bar{\nabla}_{X_j}\bar{\nabla}_{X_i}X_k - \bar{\nabla}_{[X_i, X_j]}X_k = \bar{R}_{ij}{}^l{}_k X_l.$$

This formula yields by means of the above lemma and (81)

Lemma 7.34. *The Riemannian curvature tensor of S^3_ε is given by*

$$\bar{R}(X_1,X_2)X_2 = \varepsilon^2 X_1, \quad \bar{R}(X_2,X_3)X_3 = (4-3\varepsilon^2)X_2, \quad \bar{R}(X_3,X_1)X_1 = \varepsilon^4 X_3,$$

whereas $\bar{R}(X_i,X_j)X_k = 0$ if i,j,k are all distinct. In this frame all components of $\bar{R}$ vanish up to the usual symmetries, except for

$$\bar{R}_{1212} = \varepsilon^4, \quad \bar{R}_{2323} = 4-3\varepsilon^2, \quad \bar{R}_{3131} = \varepsilon^4.$$

Proof. We compute for example

$$\begin{aligned}\bar{R}(X_1,X_2)X_2 &= \bar{\nabla}_{X_1}\bar{\nabla}_{X_2}X_2 - \bar{\nabla}_{X_2}\bar{\nabla}_{X_1}X_2 - \bar{\nabla}_{[X_1,X_2]}X_2 \\ &= 0 - \bar{\nabla}_{X_2}\bar{\nabla}_{X_1}X_2 - 2\bar{\nabla}_{X_3}X_2 = \varepsilon^2 X_1.\end{aligned}$$

Finally we determine the components of the $(4,0)$-tensor version by $\bar{R}_{ijkl} = g^\varepsilon_{km}\bar{R}_{ij}{}^m{}_l$. □

Remark 7.35. By the formula for sectional curvature

$$K(X_i,X_j) = \frac{\bar{R}_{ijij}}{g^\varepsilon_{ii}g^\varepsilon_{jj} - (g^\varepsilon_{ij})^2}$$

we obtain for the sectional curvatures

$$K(X_1,X_2) = \varepsilon^2, \quad K(X_2,X_3) = 4-3\varepsilon^2, \quad K(X_3,X_1) = \varepsilon^2.$$

Thus S^3_ε remains a space of nonnegative sectional curvature, however small ε might be chosen. Therefore, if S^3_ε is an ambient space where convex hypersurfaces do not remain convex, this must be due to the last crucial quantity in the evolution equation of the second fundamental form, the gradient of $\bar{R}$.

7.3.1 Local symmetry and distortion

As we have already seen in the last chapter, constant sectional curvature implies local symmetry, or $\bar{\nabla}\bar{R} \equiv 0$. This is for example true for the hypersphere with sectional curvatures constant 1. The notion *symmetric* stems from the fact that for every Riemannian manifold M with parallel curvature tensor at each $p \in M$ there can be found an isometry I_p defined in a neighbourhood of p, such that $DI_p = -I$ on T_pM, where I is the identity. That is, I_p is a local symmetry. Obviously, this is true for the hypersphere.

If now, however, the hypersphere gets distorted by the Berger metrics, it is obvious as well that it will not be possible to find such local symmetries – the space is not locally symmetric. In fact the gradient of $\bar{R}$ deviates from being zero more and more, the smaller ε is chosen.

Lemma 7.36. *The components of the gradient of the $(3,1)$-version of $\bar{R}$, which is a $(4,1)$-tensor, all vanish except for*

$$\bar{\nabla}_2\bar{R}_{1223} = \bar{\nabla}_3\bar{R}_{1323} = 4\varepsilon^4 - 4\varepsilon^2$$

and symmetric components.

Proof. The result is obtained by an application of the formula

$$\bar{\nabla}_m\bar{R}_{ijkl} = X_m(\bar{R}_{ijkl}) - C^s_{mi}\bar{R}_{sjkl} - C^s_{mj}\bar{R}_{iskl} - C^s_{mk}\bar{R}_{ijsl} - C^s_{ml}\bar{R}_{ijks},$$

where $\bar{\nabla}_{X_i}X_j = C^k_{ij}X_k$ according to lemma 7.33. Furthermore we have used the observation that $X_m(\bar{R}_{ijkl}) \equiv 0$ for all indices and everywhere on S^3_ε due to the left-invariance of $\bar{R}_{ijkl}$ in the frame $\{X_i\}$. □

Remark 7.37. Clearly, whenever $\varepsilon < 1$, it is $|\bar{\nabla}\bar{R}|^2 > 0$.

7.4 Constructing a hypersurface

The previous calculations have confirmed that the Berger sphere S^3_ε with $0 < \varepsilon < 1$ meets our demands on an ambient with positive sectional curvatures (remark 7.35) and a non-vanishing curvature gradient (remark 7.37).

Therefore, having the important geometric quantities at our disposal, we are now ready to construct a hypersurface $M \subset S^3_\varepsilon$ which behaves the way we want. To this end we basically follow the procedure used in the hyperbolic space example.

We start by deriving the conditions the surface M has to satisfy at some point p in order to be convex at p and, at the same time, to have the property of losing this convexity as soon as it is moved by mean curvature flow. Afterwards we show the existence of a smooth graph hypersurface around p, which is is even strictly convex in a neighbourhood of p. As a preparation for the graph construction, we investigate the geometry of a natural submanifold of S^3_ε.

Remark 7.38. Without loss of generality we may choose $p_0 = E \in S^3_\varepsilon$, where E is the neutral element in $SU(2) = S^3_\varepsilon$, that is, the identity matrix.

Remark 7.39. Since we do not construct a global hypersurface, but only a local one, interpretations in view of theorem 4.14 are not as strong as in the previous example in hyperbolic space. For further discussions we refer to the next chapter.

7.4.1 Conditions at E

For a surface M with $E \in M$ we describe sufficient conditions for the way the tangent space $T_E M$ has to be situated in $T_E S^3_\varepsilon$, such that M loses its convexity under mean curvature flow at E, provided a solution exists.

For this purpose assume that M is immersed in S^3_ε in such a way that $E \in M$. Let $e_1, e_2 \in T_E M$ be a basis of $T_E M \subset T_E S^3_\varepsilon$ and let $e_0 \in T_E M$ be a choice of a unit normal vector. Moreover let (c_i^α), $i = 0, 1, 2$, $\alpha = 1, 2, 3$ be the 3×3-matrix which represents e_i as a linear combination of the basis X_1, X_2, X_3, that is

$$e_i = c_i^\alpha X_\alpha, \quad i = 0, 1, 2.$$

Furthermore suppose that the second fundamental form h_{ij} of M satisfy $h_{11} = h_{12} = h_{21} = 0$ as well as $h_{22} > 0$ with respect to the basis e_1, e_2, whence M is convex at E with a null-eigenvector of h_{ij}. We want to force h_{ij} to immediately take a strictly negative eigenvalue under mean curvature flowin order to destroy the convexity. To this end we again invoke the Hurwitz criterion and show that $\det h_{ij}$, which vanishes at E, has a strictly negative time derivative. We have

$$\frac{\partial}{\partial t}(\det h_{ij}) = h_{11}\frac{\partial}{\partial t}h_{22} + h_{22}\frac{\partial}{\partial t}h_{11} - 2h_{12}\frac{\partial}{\partial t}h_{12} = h_{22}\frac{\partial}{\partial t}h_{11} < 0 \quad \text{at } E.$$

Due to $h_{22} > 0$ it remains to force $\frac{\partial}{\partial t}h_{11}$ to be strictly negative.

As usual we denote by $\bar{g} = g^\varepsilon$ the ambient and by g the induced metric. By means of the matrix (c_i^α) the evolution equation (3.7) for h_{11} takes the form

$$\begin{aligned} \frac{\partial}{\partial t}h_{11} &= \Delta h_{11} + 2h_{22}\bar{R}^2{}_1{}^2{}_1 - g^{lk}\left[\bar{\nabla}_1 \bar{R}_{0l1k} + \bar{\nabla}_l \bar{R}_{011k}\right] \\ &= \Delta h_{11} + 2h_{22} g^{2l} g^{2k} c_l^\alpha c_1^\beta c_k^\gamma c_1^\delta \bar{R}_{\alpha\beta\gamma\delta} \\ &- g^{lk} c_l^\alpha c_0^\beta c_l^\gamma c_1^\delta c_k^\eta \left[\bar{\nabla}_\alpha \bar{R}_{\beta\gamma\delta\eta} + \bar{\nabla}_\gamma \bar{R}_{\beta\alpha\delta\eta}\right]. \end{aligned} \tag{82}$$

Now let us consider a special case: Assume that e_1, e_2 be orthonormal, or $g_{ij} = \delta_{ij}$. Furthermore suppose that $T_E M \subset \operatorname{span}(X_1, X_2)$. Since $\bar{g}$ is orthgonal, we may choose $e_0 = X_3$ as our unit normal. Observe that the angle φ between X_1 and e_1 thoroughly determines how the e_i-frame is situated in $T_E S^3_\varepsilon$, as long as we agree that each basis describes the same orientation of $T_E M$.

Accordingly, the matrix $C = (c_i^\alpha)$ takes the form

$$C = \begin{pmatrix} 0 & 0 & 1 \\ \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}.$$

In this case the terms in (82) simplify considerably. We have

$$\begin{aligned} g^{2l} g^{2k} c_l^\alpha c_1^\beta c_k^\gamma c_1^\delta \bar{R}_{\alpha\beta\gamma\delta} &= \bar{R}_{1212}((c_2^1)^2 (c_1^2)^2 - 2c_2^2 c_1^1 c_2^1 c_1^2 + (c_2^2)^2 (c_1^1)^2) \\ &+ \bar{R}_{2323}((c_2^2)^2 (c_1^3)^2 - 2c_2^3 c_1^2 c_2^2 c_1^3 + (c_2^3)^2 (c_1^2)^2) \\ &+ \bar{R}_{3131}((c_2^3)^2 (c_1^1)^2 - 2c_2^1 c_1^3 c_2^3 c_1^1 + (c_2^1)^2 (c_1^3)^2) \\ &= (\cos^2\varphi + \sin^2\varphi)^2 \bar{R}_{1212} = \varepsilon^4 \end{aligned}$$

and

$$g^{lk} c_l^\alpha c_0^\beta c_l^\gamma c_1^\delta c_k^\eta \left[\bar{\nabla}_\alpha \bar{R}_{\beta\gamma\delta\eta} + \bar{\nabla}_\gamma \bar{R}_{\beta\alpha\delta\eta}\right] = 8\varepsilon^2(1-\varepsilon^2)\sin\varphi\cos\varphi.$$

Thus (82) simplifies to

$$\frac{\partial}{\partial t} h_{11} = \Delta h_{11} + 2h_{22}\varepsilon^4 + 8\varepsilon^2(\varepsilon^2 - 1)\sin\varphi\cos\varphi. \tag{83}$$

We see that indeed we may arrange that the right hand side of this equation gets strictly negative by choosing $\varepsilon < 1$, $\varphi \in]0, \frac{\pi}{2}[$ and h_{22}, Δh_{11} sufficiently small. As already mentioned, the Berger parameter ε must differ from 1, since otherwise we would have a locally symmetric ambient space, the hypersphere.

Remark 7.40. Observe that the crucial negative portion in equation (83) is contributed by the last term, which stems from the $\bar{\nabla}\bar{R}$-terms.

Based upon these considerations we now locally construct a graph surface satisfying all conditions we have assumed.

Definition 7.41. A (smooth) hypersurface M of an n-dimensional Riemannian manifold N is *locally represented as a graph*, if for all $p \in M$ there is a neighbourhood U of p in N along with a diffeomorphism $\phi : U \to V \subset \mathbb{R}^n$ and a smooth function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $U = \phi^{-1}(V \cap \{(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_n))\})$.

Remark 7.42. Of course every smoothly immersed hypersurface is always locally representable as a graph.

In view of this definition, the practical questions are how ϕ should be chosen, or, equivalently, which natural submanifolds of N suggest to be employed as the underlying 'base' hypersurface

$$\phi^{-1}(V \cap \{(x_1, \ldots, x_{n-1}, 0),\, x_i \in \mathbb{R}\})$$

on the one hand, and, on the other hand, as the curves

$$\phi^{-1}(V \cap \{(x_1, \ldots, x_{n-1}, s),\, s \in \mathbb{R}\}) \quad \text{for fixed } x_1, \ldots, x_{n-1}.$$

7.4.2 The (E, I, J)-equator

Inspired by the local example we have constructed in the previous section we consider the simplest hypersurface of $SU(2)$ tangent to $\mathrm{span}(X_1, X_2)$ at $E \in SU(2)$, the (E, I, J)-equator

$$S^2 = S^2_{(E,I,J)} = \mathrm{span}(E, I, J) \cap S^3 = \{(\cos t e^{i\theta}, \sin t)\} \in \mathbb{H}.$$

In the sequel we are going to illuminate the geometry of S^2 in terms of the question whether we can construct a promising graph over S^2. In other words, we are particularly interested in convexity or generally the second fundamental form.

Example 7.43. In the case that the Berger parameter is $\varepsilon = 1$, we know that great circles are exactly the geodesics in S^3. Since the metric on S^2 induced by S^3 coincides with the one induced from the ambient space $\mathbb{H} \equiv \mathbb{R}^4$, great circles are also exactly the geodesics of the isometric immersion $S^2 \subset S^3$. Thus, geodesics in S^2 are geodesic in S^3 as well - the immersion is *totally geodesic.* This, in turn, means that the second fundamental form vanishes everywhere on S^2, see for example [2], proposition 2.9.

We first need a reference frame for S^2. Considering the parametrisation

$$\Phi : (t, \theta) \mapsto (\cos t e^{i\theta}, \sin t), \quad t \in]-\pi/2, \pi/2[,\ \theta \in [-\pi, \pi[,$$

we have the natural frame at $p = \Phi(t, \theta)$

$$\begin{aligned} \xi_1(p) :&= \frac{\partial}{\partial t}(p) = (-\sin t e^{i\theta}, \cos t) \\ \xi_2(p) :&= \frac{\partial}{\partial \theta}(p) = (i \cos t e^{i\theta}, 0). \end{aligned}$$

Let us from now on use the identification

$$(\cos t e^{i\theta_1}, \sin t e^{i\theta_2}) \leftrightarrow \begin{pmatrix} \cos t e^{i\theta_1} & \sin t e^{i\theta_2} \\ -\sin t e^{-i\theta_2} & \cos t e^{-i\theta_1} \end{pmatrix}.$$

Then we may rewrite the above frame as

$$\begin{aligned} \xi_1(p) :&= -\sin t \cos\theta E - \sin t \sin\theta I + \cos t J, \\ \xi_2(p) :&= -\cos t \sin\theta E + \cos t \cos\theta I. \end{aligned} \tag{84}$$

Since all crucial geometrical quantities of S^3_ε are known with respect to the frame (X_i), $i = 1, 2, 3$, we need a representation of ξ_i in this frame.

According to example 7.22, we have at $p = \Phi(t, \theta)$

$$\begin{aligned} X_1(p) &= aI - bE - cK \\ X_2(p) &= aJ + bK - cE \\ X_3(p) &= aK - bJ + cI \end{aligned} \tag{85}$$

for $a = \cos t \cos\theta$, $b = \cos t \sin\theta$, and $c = \sin t$.

Since $\xi_i(p) \in T_p S^3_\varepsilon$, they are representable with respect to the basis $X_1(p)$, $X_2(p)$, $X_3(p)$, and the system of linear equations

$$\xi_i = b_i^\alpha X_\alpha, \quad i = 1, 2,$$

must be solvable. In fact we compute by (84) and (85) that the coefficient vectors b_1 and b_2 of ξ_1 and ξ_2 in the above basis have the form

$$b_1 = \big(0, \quad \cos\theta, \quad -\sin\theta\big), \qquad b_2 = \big(\cos^2 t, \quad \sin t\cos t\sin\theta, \quad \sin t\cos t\cos\theta\big).$$

Denoting the ith entry of these vectors by b_i^α for $i = 1, 2$, we find a normal vector field by modifying the vector product $\wedge$ according to the metric $\bar{g} = g_\varepsilon$, that is, we have a normal field $\tilde{\nu}$ given by the component vector $(\tilde{\nu}^1, \tilde{\nu}^2, \tilde{\nu}^3)$ with respect to X_1, X_2, X_3, where

$$\begin{aligned} \varepsilon^2\tilde{\nu}^1 &= b_1^2 b_2^3 - b_1^3 b_2^2 \\ \tilde{\nu}^2 &= b_1^3 b_2^1 - b_1^1 b_2^3 \\ \tilde{\nu}^3 &= b_1^1 b_2^2 - b_1^2 b_2^1. \end{aligned}$$

To be precise, we have $\tilde{\nu}_1 = \varepsilon^{-2}(\cos^2\theta\sin t\cos t + \sin^2\theta\sin t\cos t) = \varepsilon^{-2}\sin t\cos t$. Similarly, $\tilde{\nu}_2 = -\sin\theta\cos^2 t$ and $\tilde{\nu}_3 = -\cos\theta\cos^2 t$.

Observe that at $\Phi(0,0) = E$ we have $\tilde{\nu}^1 = \tilde{\nu}^2 = 0$ and $\tilde{\nu}^3 = -1$, that is, $\tilde{\nu}(E) = -X_3$. We thus reverse the sign in order to match the situation in section 7.4.1. Therefore, since $|\tilde{\nu}|^2 = \bar{g}(\tilde{\nu}, \tilde{\nu}) = \cos^2 t$, we have a unit normal field given by $\nu := -(\cos t)^{-1}\tilde{\nu}$.

Having all important quantities at our disposal, we now compute the second fundamental form h_{ij} with respect to the frame ξ_1, ξ_2. We use the Weingarten equations (6) to find

$$h_{ij} = -\bar{g}(\bar{\nabla}_{\xi_i}\xi_j, \nu) = -\bar{g}(\bar{\nabla}_{\xi_i}(b_j^\alpha X_\alpha), \nu) = -\bar{g}(\xi_i(b_j^\alpha)X_\alpha + b_j^\alpha b_i^\beta \bar{\nabla}_{X_\beta}X_\alpha, \nu).$$

If we insert the values of the connection $\bar{\nabla}$ computed in 7.33, we get for the second fundamental form at $\Phi(t, \theta)$

$$\begin{aligned} h_{ij} = -\varepsilon^2 \quad & \nu^1 \quad (\xi_i(b_j^1) + b_j^3 b_i^2 - b_j^2 b_i^3) \\ - \quad & \nu^2 \quad (\xi_i(b_j^2) + b_j^1 b_i^3 \varepsilon^2 - b_j^3 b_i^1 (2 - \varepsilon^2)) \\ - \quad & \nu^3 \quad (\xi_i(b_j^3) + b_j^2 b_i^1 (2 - \varepsilon^2) - b_j^1 b_i^2 \varepsilon^2), \end{aligned} \tag{86}$$

where all quantities have to be evaluated at (t, θ).

Remark 7.44. Here the b_i^α are interpreted as C^∞-functions on $S^2 \subset S^3_\varepsilon$, to which the vector fields ξ_k - thought of as derivations - may be applied. In particular we have

$$\xi_1(b_i^\alpha) = \frac{\partial}{\partial t} b_i^\alpha(t, \theta) \quad \text{and} \quad \xi_2(b_i^\alpha) = \frac{\partial}{\partial \theta} b_i^\alpha(t, \theta)$$

at $p = \Phi(t, \theta)$.

Since there are no further simplifications to be done (for example due to suitable symmetries), we find by a direct calculation based upon (86) that at $\Phi(t, \theta)$ we have the second fundamental form

$$h_{11} = h_{22} = 0 \quad \text{and} \quad h_{12} = h_{21} = (\varepsilon^2 - 1)\cos^3 t.$$

Remark 7.45. For reasons of completeness we mention that for $t = \pm\pi/2$ the frame ξ_1, ξ_2 degenerates. In other words, we cannot immediately read h_{ij} at the poles J and $-J$. However, if we take the frame

$$\tilde{\xi_1} := \xi_1, \quad \tilde{\xi_2} := (\cos t)^{-1}\xi_2,$$

the components of h_{ij} take the form

$$\tilde{h}_{11} = \tilde{h}_{22} = 0, \quad \tilde{h}_{12} = \tilde{h}_{21} = \cos^2 t(\varepsilon^2 - 1)$$

and we may, for fixed θ, let $t \to \pm\pi/2$ in order to observe that $|h_{ij}| \to 0$, that is, $h = 0$ at $\pm J$. This result is in accordance with the fact that $\pm J$ are the only points of $S^2_{(E,I,J)}$ whose tangent plane has no X_1-component, as one easily verifies. Consequently, in this case the distortion of X_1 by ε does not influence the fact that geodesics of $S^2_{(E,I,J)}$ starting at $\pm J$ are also geodesics in S^3_ε, and the second fundamental form vanishes (compare with example 7.43).

Since at E the (E, I, J)-equator is tangent to $\text{span}(X_1, X_2) \subset T_E S^3_\varepsilon$ it is of particular interest in view of section 7.4.1 to know h_{ij} at E with respect to $X_1(E), X_2(E)$.

For this purpose observe that at E we have $t = \theta = 0$ and

$$\xi_1(E) := J = X_2 \quad \text{and} \quad \xi_2(E) := I = X_1,$$

whence with respect to X_1, X_2 the second fundamental form takes the form

$$h_{11} = h_{22} = 0, \quad h_{12} = h_{21} = (\varepsilon^2 - 1) \quad \text{at } E.$$

Remark 7.46. By normalising and rotating the basis X_1, X_2 of $T_E S^2_{(E,I,J)}$ we can arrange that h_{ij} takes the form $\text{diag}(\kappa_1, \kappa_2)$. In this way we can read off directly whether S^2 is convex at E or not. Moreover, this is of special interest in view of section 7.4.1, where we demand h_{ij} to be diagonal at E.

Observe that by $\varepsilon^{-1}X_1$, X_2 we have given an orthonormal basis of $T_E S^2_{(E,I,J)}$. If we rotate this basis by the angle $\alpha = -\pi/4$, we obtain the basis

$$\begin{aligned} Y_1 &= \varepsilon^{-1} \sin\frac{\pi}{4} X_1 + \cos\frac{\pi}{4} X_2 = \frac{\sqrt{2}}{2}(-\varepsilon^{-1}X_1 + X_2) \\ Y_2 &= \varepsilon^{-1} \cos\frac{\pi}{4} X_1 - \sin\frac{\pi}{4} X_2 = \frac{\sqrt{2}}{2}(\varepsilon^{-1}X_1 + X_2), \end{aligned}$$

in terms of which h_{ij} takes the diagonal form $\kappa_1 = h_{11} = -(1-\varepsilon^2)/\varepsilon$, $\kappa_2 = h_{22} = (\varepsilon^2-1)/\varepsilon$, and $h_{12} = h_{21} = 0$.

Similarly one can prove

Lemma 7.47. *The principal curvatures κ_1 and κ_2 of the (E, I, J)-equator $S^2_{(E,I,J)} \subset S^3_\varepsilon$ have the form*

$$\kappa_1 = \varepsilon^{-1}(1 - \varepsilon^2)\cos^3 t \quad \textit{and} \quad \kappa_2 = \varepsilon^{-1}(\varepsilon^2 - 1)\cos^3 t$$

at $\Phi(t, \theta) \in S^2_{(E,I,J)}$.

Remark 7.48. This lemma shows that the (E, I, J)-equator $S^2_{(E,I,J)}$ is non-convex in the sense that the principal curvatures have different signs everywhere (except for the points $\pm J$), if $\varepsilon < 1$. Therefore $S^2_{(E,I,J)}$ does not satisfy the conditions of section 7.4.1.

7.4.3 Local graphs over $S^2_{(E,I,J)}$

In this section we use the following approach: It is possible to construct (local) graph hypersurfaces of S^2 by assigning an angle $\gamma(p)$ to each point $p \in S^1$ of some great circle S^1 and then moving p along the great circle through p perpendicular to S^1 by the angle $\gamma(p)$. This method is generalized to graphs in S^3_ε over $S^2_{(E,I,J)}$ if we define

$$F(t,\theta) := \cos(\gamma(t,\theta))(\cos t(\cos\theta E + \sin\theta I) + \sin t J) + \sin(\gamma(t,\theta))K. \tag{87}$$

If we suppose that γ is a smooth function with values in $]-\pi/2, \pi/2[$, F is a diffeomorphism for $t \in]-\pi/2, \pi/2[$ and $\theta \in [-\pi, \pi[$. We denote the hypersurface parametrized in this fashion by M. M can be thought of as a graph surface over $S^2_{(E,I,J)}$, where the graph fibers are the great circles perpendicular to $S^2_{(E,I,J)}$.

We derive the conditions the graph function γ has to satisfy at E in order to match the assumptions of section 7.4.1 on the one hand, and, on the other hand, to guarantee that M is strictly convex in a neighbourhood around E.

In section 7.4.1 we have assumed $T_E M = \text{span}(X_1, X_2)$. Since $T_E S^2_{(E,I,J)} = \text{span}(X_1, X_2)$ we clearly must set

$$\gamma(0,0) = 0 \quad \text{and} \quad \frac{\partial\gamma}{\partial t}(0,0) = \frac{\partial\gamma}{\partial\theta}(0,0) = 0 \tag{88}$$

to make this assumption valid. We therefore assume these conditions to be satisfied everywhere in the remaining text. We then have the follwing

Lemma 7.49. *Let $M \subset S^3_\varepsilon$ be the hypersurface described by the graph of γ as above. Suppose $T_E M = span(X_1, X_2)$. With respect to the unit normal $\nu = X_3$ at E, the second fundamental form of M at E takes the form*

$$h_{11} = -\frac{\partial^2\gamma}{\partial t^2}(0,0), \quad h_{22} = -\frac{\partial^2\gamma}{\partial\theta^2}(0,0) \quad h_{12} = h_{21} = -1 + \varepsilon^2 - \frac{\partial^2\gamma}{\partial t\partial\theta}(0,0)$$

with respect to $\xi_1 = X_2$, $\xi_2 = X_1 \in T_E M$.

Proof. Let the coordinate vector fields given by

$$\xi_1(p) := \frac{\partial F}{\partial t}(p) \quad \text{and} \quad \xi_2(p) := \frac{\partial F}{\partial\theta}(p) \quad \text{at } p = F(t,\theta) \in M$$

be our reference frame, which we want to express in terms of the left-invariant basis $\{X_i\}$. For this purpose we invoke example 7.22, showing that we have at $p = F(t,\theta)$

$$\begin{aligned} X_1(p) &= aI - bE - cK + dJ \\ X_2(p) &= aJ + bK - cE - dI \\ X_3(p) &= aK - bJ + cI - dE \end{aligned} \tag{89}$$

with $a = \cos(\gamma(t,\theta))\cos t\cos\theta$, $b = \cos(\gamma(t,\theta))\sin\theta$, $c = \cos(\gamma(t,\theta))\sin t$, and $d = \sin(\gamma(t,\theta))$, where the choice of coefficients follows from (87).

If we write down the ξ_i in terms of E, I, J, K, we can express the inner frame $\{\xi_i\}$ in terms of the ambient frame $\{X_\alpha\}$. As one easily verifies, we get the expressions

$$\xi_i = b_i^\alpha X_\alpha \quad \text{for } i = 1, 2$$

with

$$\begin{aligned}
b_1^1 &= -\frac{\partial\gamma}{\partial t}\sin t + \cos t\sin\gamma\cos\gamma \\
b_1^2 &= \frac{\partial\gamma}{\partial t}\cos t\sin\theta + \cos\theta\cos^2\gamma + \sin t\sin\theta\cos\gamma\sin\gamma \\
b_1^3 &= \frac{\partial\gamma}{\partial t}\cos t\cos\theta - \sin\theta\cos^2\gamma + \sin t\cos\theta\cos\gamma\sin\gamma \\
b_2^1 &= -\frac{\partial\gamma}{\partial\theta}\sin t + \cos^2 t\cos^2\gamma \\
b_2^2 &= \frac{\partial\gamma}{\partial\theta}\cos t\sin\theta + \cos t\sin t\sin\theta\cos^2\gamma - \cos t\cos\theta\cos\gamma\sin\gamma \\
b_2^3 &= \frac{\partial\gamma}{\partial\theta}\cos t\cos\theta + \cos t\sin t\cos\theta\cos^2\gamma + \cos t\sin\theta\cos\gamma\sin\gamma.
\end{aligned}$$

Taking into account (88) along with $t = \theta = 0$ at E it follows in particular that we have

$$\xi_1 = X_2 \quad \text{and} \quad \xi_2 = X_1 \quad \text{at } E,$$

which allows us to choose $\nu := X_3$ as the unit normal at E.

We now compute h_{ij} according to (86). However, this formula demands that the derivatives of the b_i^α's be computed. These terms get very lengthy and reasonable simplifications cannot be done; we therefore restrict ourselves to displaying the results without showing the computational details. Nevertheless it is straightforward to verify the calculations. We have for $t = \theta = 0$

$$\begin{aligned}
\xi_1(b_1^3) = \frac{\partial}{\partial t}b_1^3 &= \frac{\partial^2\gamma}{\partial t^2}(0,0), \quad \xi_2(b_2^3) = \frac{\partial}{\partial\theta}b_2^3 = \frac{\partial^2\gamma}{\partial\theta^2}(0,0), \\
\xi_1(b_2^3) = \frac{\partial}{\partial t}b_2^3 &= 1 + \frac{\partial^2\gamma}{\partial t\partial\theta}(0,0), \quad \xi_2(b_1^3) = \frac{\partial}{\partial\theta}b_1^3 = -1 + \frac{\partial^2\gamma}{\partial\theta\partial t}(0,0),
\end{aligned} \tag{90}$$

whereas all other derivatives vanish.

Since $b_i^\alpha = 0$ at E except for $b_1^2 = b_2^1 = 1$, and since $\nu^1 = \nu^2 = 0$ and $\nu^3 = 1$, the formula (86) yields the assertion. □

Our aim is now to choose the first derivatives of γ in such a way that we get the situation described in section 7.4.1. In particular we have to determine the angle φ by which the basis making h_{ij} diagonal is rotated with respect to the above basis ξ_1, ξ_2. For this purpose we provide the following

Lemma 7.50. *Let $h : T_EM \times T_EM \to \mathbb{R}$ be the second fundamental form of M at E. Let h_{ij} be its components in terms of a basis $\{e_i\}$, $i = 1, 2$, which is orthonormal with respect to some inner product g. Furthermore let A be the corresponding self-adjoint endomorphism $T_EM \to T_EM$ with eigenvalues κ_1, κ_2 belonging to the eigenvectors v_1 and v_2, respectively.*

Suppose that $h_{11} > 0$, $h_{12} = h_{21} < 0$ and $\det h_{ij} = h_{11}h_{22} - h_{12}^2 = 0$.

Then we have $\kappa_1 = 0$ and $\kappa_2 = h_{11} + h_{22} > 0$ and the angle φ between v_1 and e_1 given by

$$v_1 = (\cos\varphi e_1 + \sin\varphi e_2)||v_1||$$

satisfies

$$\varphi = \arccos\frac{h_{11}}{\sqrt{h_{12}^2 + h_{11}^2}} \in]0, \pi/2[.$$

Proof. Since $\{e_i\}$ is orthonormal, A is represented by the matrix

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$

Then the characteristic polynomial p_A of A has the form

$$p_A(\kappa) = \kappa^2 - \kappa(h_{11} + h_{22}) + h_{11}h_{22} - h_{12}^2 = \kappa^2 - \kappa(h_{11} + h_{22}),$$

and the equation $p_A(\kappa) = 0$ has the solutions $\kappa_1 = 0$ and $\kappa_2 = h_{11} + h_{22}$. From $h_{22} = h_{12}^2/h_{11} > 0$ it follows $\kappa_2 > 0$.

Moreover, the eigenspace corresponding to $\kappa_1 = 0$ is given by $\mathbb{R}v_1$, where

$$v_1 = \begin{pmatrix} h_{12}^2/h_{11} \\ -h_{12} \end{pmatrix}.$$

Since $-h_{12} > 0$, both components of v_1 are strictly positive. Thus $\varphi \in]0, \pi/2[$. Furthermore we have

$$\cos\varphi = \frac{g(v_1, e_1)}{\sqrt{g(v_1, v_1)g(e_1, e_1)}} = \frac{h_{11}}{\sqrt{h_{11}^2 + h_{12}^2}}.$$

□

In order to apply this lemma in the case of the tangent space T_EM we first represent h_{ij} in terms of the orthonormal basis $e_1 = \varepsilon^{-1}X_1$, $e_2 = X_2$ of T_EM, which is according to lemma 7.49

$$h_{11} = -\varepsilon^{-2}\frac{\partial^2\gamma}{\partial\theta^2}, \quad h_{22} = -\frac{\partial^2\gamma}{\partial t^2}, \quad h_{12} = h_{21} = \varepsilon^{-1}(-1 + \varepsilon^2 - \frac{\partial^2\gamma}{\partial t\partial\theta}).$$

Then it is evident that by a suitable choice of the second derivatives of γ (which may depend on ε) we can arrange that the conditions of the lemma are satisfied. Furthermore we may clearly force φ to take any value in $]0, \pi/2[$.

Observe that if the lemma applies we automatically have the situation described in section 7.4.1. Starting with the above orthonormal basis e_1, e_2 we rotate it by an angle $\varphi \in]0, \pi/2[$ in order to find the basis v_1, v_2 in terms of which h_{ij} gets diagonal. Recall that in this case we have found that with respect to v_1, v_2 the evolution equation of h_{11} at E takes the form

$$\frac{\partial}{\partial t}h_{11} = \Delta h_{11} + 2h_{22}\varepsilon^4 + 8\varepsilon^2(\varepsilon^2 - 1)\sin\varphi\cos\varphi, \tag{91}$$

which we want to take a negative value.

As the last term in (91) contributes a strictly negative value, we have to keep Δh_{11} and h_{22} small enough. The Laplacian of h_{11} involves third partial derivatives of γ and will be cared for later on. However, the following example demonstrates that it indeed is possible to find a good choice for ε and the second partial derivatives of γ at E satisfying the conditions of lemma 7.50 of γ at E, which makes the whole right hand side negative, *while neglecting the Laplacian term for the moment.*

Example 7.51. If we set at E

$$\varepsilon = \frac{1}{2}, \quad \partial_{tt}^2\gamma = -\frac{1}{4}, \quad \partial_{\theta\theta}^2\gamma = -\frac{1}{16}, \quad \partial_{t\theta}^2\gamma = -\frac{5}{8},$$

we have in terms of the orthogonal basis $\varepsilon^{-1}X_1$, X_2

$$h_{11} = -\varepsilon^{-2}\partial^2_{\theta\theta}\gamma = \frac{1}{4} > 0 \quad \text{and} \quad h_{12} = \frac{-1+\varepsilon^2-\partial^2_{t\theta}\gamma}{\varepsilon} = -\frac{1}{4} < 0,$$

and since $h_{22} = -\partial^2_{tt}\gamma = 1/4$ we have $h_{11}h_{22} - h_{12}^2 = 0$. Thus the conditions of lemma 7.50 are satisfied and we conclude

$$\cos\varphi = \frac{h_{11}}{\sqrt{h_{11}^2 + h_{12}^2}} = \frac{1}{2}\sqrt{2},$$

yielding $\varphi = \pi/4$ according to the lemma. Moreover it follows $\kappa_1 = 0$, $\kappa_2 = h_{11} + h_{22} = 1/2$. With $\cos\varphi\sin\varphi = 1/2$ we find that the right hand side of (91) without the Laplacian gets strictly negative, according to

$$2\kappa_2\varepsilon^4 + 8\varepsilon^2(\varepsilon^2-1)\sin\varphi\cos\varphi = \frac{1}{16} - \frac{3}{4} = -\frac{11}{16} < 0.$$

Having seen that our graph approach indeed admits choices of γ which imply the desired behaviour of M at E, there now remain two questions to be checked:

Firstly, whether we can extend γ and thus M around E (at least locally), such that M is strictly convex away from E.

Secondly, whether the Laplacian of κ_1 at E can be made sufficiently small in order to ensure that we indeed have $\frac{\partial}{\partial t}h_{11} < 0$ in (91).

7.4.4 Conditions involving higher derivatives of γ

These two questions will lead to conditions involving third and fourth derivatives of γ. Since expressions will get quite complicated at some stages, we will not always perform calculations up to the last detail; we rather discuss existence aspects instead of giving explicit examples.

First we provide some useful results.

Lemma 7.52. *Under the assumption* (88) *the induced connection* ∇ *of* M *given by the Christoffel symbols* Γ^k_{ij} *gets trivial at* E*, that is,* $\Gamma^k_{ij} = 0$ *at* E*.*

Proof. It is a fact from differential geometry that the induced connection ∇ of M given by the Christoffel symbols Γ^k_{ij}, which base on the induced metric g_{ij}, can also be obtained by projecting the ambient connection $\bar\nabla$ onto TM. Concretely, if $\pi^\top$ is the projection of $T_pS^3_\varepsilon$ onto T_pM along the orthogonal complement of T_pM, it is

$$\nabla_{\xi_i}\xi_j = \pi^\top(\bar\nabla_{\tilde\xi_i}\tilde\xi_j),$$

where the $\tilde\xi_i$ are smooth local extensions of the ξ_i. The equation is independent of the actual choice of the extensions.

We have $\xi_1 = X_2$ and $\xi_2 = X_1$ at E. Moreoevr, according to lemma 7.33, none of the $\bar\nabla_{X_i}X_j$ for $i,j \in \{1,2\}$ have any components in $\operatorname{span}(X_1,X_2) = T_EM$. Thus the projection vanishes and it follows

$$\nabla_{\xi_i}\xi_j = 0 \quad \text{for } i,j \in \{1,2\},$$

whence the Christoffel symbols vanish, too. □

As a consequence we get a simple formula for the Laplacian of h_{ij} at E.

Lemma 7.53. *It is*

$$\Delta h_{ij} = g^{ll}\left[\xi_l(\xi_l(h_{ij})) - \xi_l(\Gamma^m_{li})h_{mj} - \xi_l(\Gamma^m_{lj})h_{im}\right] \quad \text{at } E,$$

where all quantities are expressed in terms of $\{\xi_i\}$.

Proof. It is $\Delta h_{ij} = g^{lk}\nabla_l\nabla_k h_{ij}$. According to the definition of covariant differentiation of general tensors we have

$$\nabla_l\nabla_k h_{ij} = \xi_l(\xi_k(h_{ij})) - \xi_l(\Gamma^m_{ki})h_{mj} - \xi_l(\Gamma^m_{kj})h_{im} + \text{terms involving } \Gamma^k_{ij}.$$

The terms involving Γ^k_{ij} vanish at E due to the lemma above. The assertion follows since $g_{ij} = 0$ at E for $i \neq j$. □

Remark 7.54. Despite of the considerable simplification of the formula for Δh_{ij}, there still are derivatives of Christoffel symbols involved as well as second derivatives of h_{ij}. These terms get very complicated and we therefore restrict ourselves to the question of existence rather than giving explicit examples.

Example 7.55. We continue with example 7.51. We have found a choice of the second derivatives of γ at E such that there is an orthonormal basis e_1, e_2 of $T_E M$ in terms of which we have $h_{11} = 0$ and $h_{22} > 0$, as well as $h_{12} = h_{21} = 0$.

We now may additionally assume that we have normal coordinates $\tilde{e}_i$ at E with $e_i(E) = \tilde{e}_i(E)$. Since the Laplacian of h_{ij} is tensorial in both indices, we may simplify the computation of Δh_{11} by means of these coordinates. Similarly to the above lemma we get

$$\Delta h_{11} = g^{ll}\left[\tilde{e}_l(\tilde{e}_l(h_{11})) - \tilde{e}_l(\Gamma^m_{l1})h_{m1} - \tilde{e}_l(\Gamma^m_{l1})h_{1m}\right] \quad \text{at } E,$$

where all quantities are meant in terms of $\tilde{e}_1$, $\tilde{e}_2$. Due to $h_{11} = h_{12} = 0$, this simplifies to

$$\Delta h_{11} = \tilde{e}_1(\tilde{e}_1(h_{11}) + \tilde{e}_2(\tilde{e}_2(h_{11}) \quad \text{at } E.$$

We thus will have succeeded if we manage to arrange that $\tilde{e}_1(\tilde{e}_1(h_{11}) + \tilde{e}_2(\tilde{e}_2(h_{11}) < 11/16$ according to example 7.51.

Using the coordinates chosen in the example we now turn to the last condition we want γ to satisfy: We want to guarantee convexity of M at least in some neighbourhood of E.

For this purpose first observe that $h_{22} > 0$ at E remains so in a neighbourhood U_1 of E due to continuity reasons..

Convexity of M is equivalent to h_{ij} being positive definite. According to the Hurwitz criterion, this, in turn, is equivalent to $h_{22} > 0$ and $\det h_{ij} = h_{11}h_{22} - (h_{12})^2 > 0$. We have to care for the latter condition. At E it is $\det h_{ij} = 0$. We therefore force $\det h_{ij}$ to have a local minimum at E, which is the case if at E the conditions

$$\nabla_{\tilde{e}_k}(\det h_{ij}) = \tilde{e}_k(\det h_{ij}) = 0 \quad \Longleftrightarrow \quad \tilde{e}_k(h_{11}) = 0 \quad \text{for } k = 1, 2 \tag{92}$$

and

$$\nabla_{\tilde{e}_k}\nabla_{\tilde{e}_l}(\det h_{ij}))v^k v^l > 0 \quad \text{for all vectors } v = \{v^k\} \in T_E M \tag{93}$$

are satisfied.

As to the latter one, due to $h_{11} = h_{12} = 0$ at E we have

$$\begin{aligned} & \nabla_{\tilde{e}_k}\nabla_{\tilde{e}_l}(\det h_{ij})) = \tilde{e}_k(\tilde{e}_l(\det h_{ij})) \\ = \; & h_{22}\tilde{e}_k(\tilde{e}_l(h_{11})) + \tilde{e}_k(h_{11})\tilde{e}_l(h_{22}) + \tilde{e}_l(h_{11})\tilde{e}_k(h_{22}) - 2\tilde{e}_k(h_{12})\tilde{e}_l(h_{12}), \end{aligned}$$

and if we assume condition (92) to be valid, we have

$$\tilde{e}_k(\tilde{e}_l(\det h_{ij})) = h_{22}\tilde{e}_k(\tilde{e}_l(h_{11})) - 2\tilde{e}_k(h_{12})\tilde{e}_l(h_{12}).$$

Assume additionally $\tilde{e}_k(h_{12}) = 0$ for $k = 1, 2$. Then (93) is satisfied as long as $\tilde{e}_k(\tilde{e}_l(h_{11})) > 0$ for $k, l \in \{1, 2\}$. Observe that this is in accordance with example 7.55 in that it is still possible to keep Δh_{11} small enough to guarantee $\frac{\partial}{\partial t} h_{11} < 0$.

Accordingly, we have to show that it is possible to choose the third and fourth partial derivatives of γ at E in such a way that in addition to the conditions for the second partial derivatives of γ derived in example 7.51 we have at E

$$\tilde{e}_k(h_{11}) = 0 \quad \text{and} \quad \tilde{e}_k(h_{12}) = 0 \quad \text{for } k = 1, 2 \tag{94}$$

and

$$\tilde{e}_k(\tilde{e}_l(h_{11})) > 0 \quad \text{for } k, l \in \{1, 2\} \quad \text{and} \quad \tilde{e}_1(\tilde{e}_1(\kappa_1)) + \tilde{e}_2(\tilde{e}_2(\kappa_1)) < 11/16. \tag{95}$$

For this purpose first observe that in a neighbourhood of E we can express the frame $\{\tilde{e}_i\}$ in terms of the coordinate frame $\{\xi_i\}$ by means of the matrix A^i_j, whose components smoothly depend on the point p:

$$\tilde{e}_i(p) = A^i_j(p)\xi_j(p)$$

At E we then have

$$\begin{aligned} \tilde{e}_k(h(\tilde{e}_i, \tilde{e}_j)) &= \nabla_{\tilde{e}_k} h(\tilde{e}_i, \tilde{e}_j) = \nabla_{(A^n_k \xi_n)} h(A^l_i \xi_l, A^m_j \xi_m) = A^n_k \nabla_{\xi_n}(A^l_i A^m_j h(\xi_l, \xi_m)) \\ &= A^n_k A^l_i A^m_j \nabla_{\xi_n} h(\xi_l, \xi_m) + A^n_k \xi_n(A^l_i A^m_j) h(\xi_l, \xi_m) \\ &= A^n_k A^l_i A^m_j \xi_n(h(\xi_l, \xi_m)) + P(A^i_j, \xi_i(A^k_l), h_{ij}), \end{aligned}$$

where P is a quantity only depending on A^i_j, $\xi_i(A^k_l)$ and h_{ij}. It is easy to verify that these quantities involve derivatives of γ only up to degree 2 and thus are already fixed according to example 7.51.

The terms $\xi_n(h(\xi_l, \xi_m))$, however, take at E the simple form

$$\xi_n(h(\xi_l, \xi_m)) = -\xi_n(\xi_l(\xi_m(\gamma))),$$

which can be verified by a direct calculation based upon equation (86). It is therefore evident that there is a suitable choice of the four different third derivatives of γ at E satisfying the four conditions of (94).

Furthermore, the second derivatives of h_{ij} at E are all expressions of the form

$$\xi_q(\xi_n(h(\xi_l, \xi_m))) = -\xi_q(\xi_n(\xi_l(\xi_m(\gamma)))) + \text{ lower order terms},$$

as a direct calculation shows. Then, similarly as above, it follows that condition (95) can be satisfied by a suitable choice of the fourth derivatives of γ at E.

Thus we have seen that it is possible to ensure the desired behaviour of M in a neighbourhood of E by knowing the Taylor series of γ up to fourth derivatives at E, which is summarized in the following

Theorem 7.56. *Let M be a smooth hypersurface given as a graph γ over $S^2_{(E,I,J)} \subset S^3_{1/2}$. Assume $\gamma(0,0) = 0$ and suppose that the partial derivatives of γ at $(0,0)$ are fixed up to fourth degree according to our considerations above.*

Then M is convex in a neighbourhood around E. Furthermore, M loses its convexity as soon as it is moved by mean curvature flow.

Remark 7.57. A next step would be to find an explicit global example for γ and, consequently, for the whole hypersurface M. The existence of such a function γ is very likely. However, it is rather difficult to explicitly construct such an example, as the terms involved get huge. For example, when trying to satisfy the assumptions made in section 7.4.1 we cannot simplify things as in the last chapter by directly working with a graph function γ which only depends either on t or on θ. Rather we need the possibility of suitably fixing all three different partial derivatives of $\gamma(t, \theta)$. One thus has to find other tools in order to at least show the existence of an extension of our local hypersurface to a smooth global closed surface. This, however, would go beyond the scope of this work.

8 Conclusions and interpretations

In the following we draw basic conclusions from the results of the last two chapters and give a brief discussion about possible interpretations.

The last two chapters have shown examples where - in contrast to Euclidean space - in general Riemannian manifolds proper convexity is *not preserved* under mean curvature flow. In view of the terms occurring in the evolution equation for the second fundamental form which involve the ambient curvature and its gradient, it is fair to expect this result. The evolution equation, namely, governs the behaviour of h_{ij} under mean curvature flow, which, in turn, determines if a surface is convex or not. Therefore it is suggested that both the ambient curvature and its gradient might work against the preservation of convexity by influencing the evolution equation in a bad way.

By separately considering two examples with different ambient spaces we additionally have shown that indeed each of these quantities - both the ambient curvature and its gradient - can destroy original convexity of the surface.

In view of the question if the convexity theorem we have proved in chapter 4 can be improved or not, these considerations strongly suggest that a crucial improvement is not possible. In other words, we have always to assume that the ambient space has bounded curvature and curvature gradient, and it is inevitable to take these bounds K_1 and L into account when looking for a modified convexity condition to be preserved.

The hyperbolic space example immediately refers to theorem 4.14 by providing a closed hypersurface. The Berger sphere example, however, has the disadvantage of only yielding a local description of a convex hypersurface, and we do not know whether it indeed can be extended to a smooth closed hypersurface. On the other hand, it allows the conclusion that if there was a *local* version of theorem 4.14, or, alternatively, if there was a proof only using local arguments, we then definitely would have to include the bound L. Observe that the proof presented in chapter 4 involves the fact that M is boundary-free to make the maximum principle apply. This is, however, the only argument referring to global properties of the evolving surface. It is therefore fair to assume that there is another proof only based on local arguments - possibly under slightly different circumstances.

9 Appendix

We give the detailed proof of lemma 5.13.

Proof. We proceed in three steps. In the first step we derive an inequality for the Laplacian of f_σ, which we integrate in the second step. We integrate by parts and estimate each term. In the third step we complete the proof by employing relation (35) and the Young inequality.

Step 1:
We derive an inequality for Δf_σ. We already have computed Δf_σ in (41). Moreover, it follows from (43)

$$\begin{aligned} & \frac{2(\alpha-1)}{H}\langle\nabla H, \nabla f_\sigma\rangle \\ = & \frac{2(\alpha-1)}{H^{\alpha+1}}\left\langle\nabla|A|^2, \nabla H\right\rangle - \frac{2\alpha(\alpha-1)}{H^{\alpha+2}}|A|^2|\nabla H|^2 + \frac{2(1-\alpha)(2-\alpha)}{nH^\alpha}|\nabla H|^2. \end{aligned}$$

Adding this equation to (41) and reorganizing terms yields

$$\begin{aligned} \Delta f_\sigma = & \underbrace{\frac{H\Delta|A|^2-\alpha|A|^2\Delta H}{H^{\alpha+1}} - \frac{2-\alpha}{n}H^{1-\alpha}\Delta H}_{[1]} \underbrace{-\frac{2}{H^{\alpha+1}}\left\langle\nabla|A|^2, \nabla H\right\rangle}_{[2]} \\ + & \frac{(1-\alpha)(2-\alpha)}{nH^\alpha}|\nabla H|^2 + \alpha(\alpha+1)\frac{|A|^2}{H^{\alpha+2}}|\nabla H|^2 \\ - & \frac{2\alpha(\alpha-1)}{H^{\alpha+2}}|A|^2|\nabla H|^2 - \frac{2(\alpha-1)}{H}\langle\nabla H, \nabla f_\sigma\rangle \end{aligned} \tag{96}$$

The intention is to avoid terms involving $|A|^2$, $\nabla|A|^2$, or $\Delta|A|^2$, since the desired estimate only contains f_σ, H, and their respective gradients.

For this purpose we rewrite $\Delta|A|^2$ by means of relation (9), which is a corollary of Simon's identity for the Laplacian of h_{ij}. However, we are not interested in how exactly the ambient curvature terms contribute to $\Delta|A|^2$, and we may weaken the information of (9) to the estimate

$$\frac{1}{2}\Delta|A|^2 \geq \langle h_{ij}, \nabla_i\nabla_j H\rangle + |\nabla_k h_{ij}|^2 - |A|^4 + Hh^{ij}h_{il}h^l_j - CH^2 - C,$$

where C depends on the bounds K_1, K_2, and L of the ambient curvature and the dimension n. For reasons of simplicity we comprise the quantities which contain h_{ij} and *no* second derivatives by

$$Z := Hh^{ij}h_{il}h^l_j - |A|^4.$$

If we apply the above estimate to (96), we obtain that expression [1] of (96) is less than or equal to

$$\begin{aligned} & \frac{2}{H^\alpha}\left\langle\tilde{h}_{ij}, \nabla_i\nabla_j H\right\rangle + \frac{2}{H^\alpha}|\nabla_k h_{ij}|^2 + \frac{2}{H^\alpha}Z \\ + & \underbrace{\left(\frac{2}{nH^{\alpha-1}} - \frac{2-\alpha}{nH^{\alpha-1}} - \frac{\alpha}{H^{\alpha+1}}|A|^2\right)}_{-\alpha/H\cdot f_\sigma}\Delta H - \frac{CH^2+C}{H^\alpha}. \end{aligned} \tag{97}$$

Observe that here we have replaced h_{ij} by the traceless second fundamental form $\tilde{h}_{ij} := h_{ij} - H/n \cdot g_{ij}$ via the relation

$$\langle h_{ij}, \nabla_i \nabla_j H \rangle = \left\langle \tilde{h}_{ij}, \nabla_i \nabla_j H \right\rangle + \frac{H}{n} \Delta H,$$

a step which is due to the fact that $|\tilde{h}_{ij}|$ can easily be estimated, as we will see.

Moreover, if we rewrite expression [2] of (96) by means of lemma 5.5 and combine this with (97) and (96), we finally obtain the inequality

$$\begin{aligned} \Delta f_\sigma \ &\geq \ \frac{2}{H^\alpha} \left\langle \tilde{h}_{ij}, \nabla_i \nabla_j H \right\rangle + \frac{2}{H^\alpha} Z - \frac{\alpha}{H} f_\sigma \Delta H + \frac{2}{H^{\alpha+2}} |H \nabla_i h_{kl} - h_{kl} \nabla_i H|^2 \\ &+ \ \frac{(2-\alpha)(\alpha-1)}{H^2} f_\sigma |\nabla H|^2 - \frac{2(\alpha-1)}{H} \langle \nabla H, \nabla f_\sigma \rangle - \frac{CH^2 + C}{H^\alpha}. \end{aligned}$$

Omitting the nonnegative terms of the right hand side eventually completes step 1 and yields the inequality

$$\Delta f_\sigma \geq \frac{2}{H^\alpha} \left\langle \tilde{h}_{ij}, \nabla_i \nabla_j H \right\rangle + \frac{2}{H^\alpha} Z - \frac{\alpha}{H} f_\sigma \Delta H - \frac{2(\alpha-1)}{H} \langle \nabla H, \nabla f_\sigma \rangle - \frac{CH^2 + C}{H^\alpha}. \quad (98)$$

Note that now there are only terms occurring which involve H, f_σ and their respective first and second derivatives. Z, however, is a quantity still involving h_{ij}, which we will care for later on.

Step 2:
We multiply (98) by f_σ^{p-1} and integrate. Since M_t has no boundary, there are no boundary terms occurring if we perform integration by parts. In particular we will demonstrate this for the most important terms of (98). We have

$$\int\limits_{M_t} f_\sigma^{p-1} \Delta f_\sigma \, d\mu = -(p-1) \int\limits_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 \, d\mu \quad (99)$$

and

$$\int\limits_{M_t} \frac{f_\sigma^p}{H} \Delta H \, d\mu = - \int\limits_{M_t} p \frac{f_\sigma^{p-1}}{H} \langle \nabla f_\sigma, \nabla H \rangle \, d\mu + \int\limits_{M_t} \frac{f_\sigma^p}{H^2} |\nabla H|^2 \, d\mu. \quad (100)$$

Furthermore, we compute by means of the Codazzi equations (lemma 2.5)

$$\begin{aligned} & g^{ik} g^{jl} \nabla_k \left(\tilde{h}_{ij} \right) \nabla_l H = g^{ik} g^{jl} \nabla_l H \nabla_k h_{ij} - \frac{1}{n} g^{ik} g^{jl} \nabla_l H \nabla_k (H g_{ij}) \\ = \ & g^{ik} g^{jl} \nabla_l H (\nabla_j h_{ik} + \bar{R}_{0ijk}) - \frac{1}{n} g^{ik} g^{jl} g_{ij} \nabla_l H \nabla_k H = |\nabla H|^2 + g^{jl} \nabla_l H \bar{R}_{0sj}{}^s - \frac{1}{n} |\nabla H|^2 \end{aligned}$$

and thus get

$$\begin{aligned} & \int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha} \left\langle \tilde{h}_{ij}, \nabla_i \nabla_j H \right\rangle d\mu = - \int\limits_{M_t} g^{ik} g^{jl} \nabla_k \left(\frac{f_\sigma^{p-1}}{H^\alpha} \tilde{h}_{ij} \right) \nabla_l H \, d\mu \\ = \ & -(p-1) \int\limits_{M_t} \frac{f_\sigma^{p-2}}{H^\alpha} \left\langle \tilde{h}_{ij}, \nabla_i f \nabla_j H \right\rangle d\mu + \alpha \int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^{\alpha+1}} \left\langle \tilde{h}_{ij}, \nabla_i H \nabla_j H \right\rangle d\mu \\ - \ & \frac{n-1}{n} \int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha} |\nabla H|^2 \, d\mu - \int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha} g^{jl} \nabla_l H \bar{R}_{0sj}{}^s \, d\mu. \end{aligned} \quad (101)$$

Using (99),(100), and (101), we rearrange terms and derive from (98) the inequality

$$
\begin{aligned}
0 \quad \geq \quad & \underbrace{(p-1)\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu}_{[1]} + \int\limits_{M_t}\frac{2}{H^\alpha}Zf_\sigma^{p-1}\,d\mu + \underbrace{(\alpha p-2\alpha+2)\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H}\left\langle\nabla f_\sigma,\nabla H\right\rangle d\mu}_{[2]}\\
+ \quad & \underbrace{2\alpha\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H^{\alpha+1}}\left\langle\tilde{h}_{ij},\nabla_i H\nabla_j H\right\rangle d\mu}_{[3]} - \underbrace{\frac{2(n-1)}{n}\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\,d\mu - \alpha\int\limits_{M_t}\frac{f_\sigma^{p}}{H^2}|\nabla H|^2\,d\mu}_{[4]}\\
- \quad & \underbrace{2(p-1)\int\limits_{M_t}\frac{f_\sigma^{p-2}}{H^\alpha}\left\langle\tilde{h}_{ij},\nabla_i f\nabla_j H\right\rangle d\mu}_{[5]}\\
- \quad & \underbrace{2\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H^\alpha}g^{jl}\nabla_l H\bar{R}_{0sj}{}^{s}\,d\mu}_{[6]} - \underbrace{\int\limits_{M_t} f_\sigma^{p-1}\frac{CH^2+C}{H^\alpha}\,d\mu}_{[7]}\,. \qquad (102)
\end{aligned}
$$

Now from $0 < f_\sigma \leq H^{2-\alpha}$ and $\alpha \leq 2$ it follows that the absolute value of [4] may be estimated from above by

$$4\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\,d\mu.$$

To estimate the terms involving $\tilde{h}_{ij}$ we utilise that $|\tilde{h}_{ij}|^2 = |A|^2 - H^2/n = f_\sigma H^\alpha \leq H^2$. From this we get by the Cauchy-Schwarz inequality that the absolute value of [3] is bounded from above by

$$4\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H^{\alpha+1}}\underbrace{|\tilde{h}_{ij}|}_{\leq H}\underbrace{|\nabla_i H\nabla_j H|}_{=|\nabla H|^2}\,d\mu \leq 4\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\,d\mu.$$

Both term [2] and term [5] involve a product of ∇f_σ and ∇H. Since we want to control the amount by which each quantity contributes to the estimate to derive, Chauchy's inequality with η (53) is a good means. We use it to estimate the absolute value of term [2] from above by

$$2(p-1)\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H}|\nabla f_\sigma||\nabla H|\,d\mu \leq (p-1)\left[\eta\int\limits_{M_t}\frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\,d\mu + \frac{1}{\eta}\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu\right]. \qquad (103)$$

Here we have used $f_\sigma \leq H^{2-\alpha}$ and the choice $a = f_\sigma|\nabla H|/H$ and $b = |\nabla f_\sigma|$ in (53).

Finally, setting $a = |\tilde{h}_{ij}||\nabla H|/H^\alpha$ and $b = |\nabla f_\sigma|$ in (53), we estimate the absolute value of term [5] from above by

$$\begin{aligned} & 2(p-1)\int\limits_{M_t} \frac{f_\sigma^{p-2}}{H^\alpha}|\tilde{h}_{ij}||\nabla f_\sigma||\nabla H|\,d\mu \\ \leq\ & (p-1)\left[\eta\int\limits_{M_t} \frac{f_\sigma^{p-2}}{H^{2\alpha}}|\tilde{h}_{ij}|^2|\nabla H|^2\,d\mu + \frac{1}{\eta}\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu\right] \\ =\ & (p-1)\left[\eta\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^{\alpha}}|\nabla H|^2\,d\mu + \frac{1}{\eta}\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu\right], \end{aligned}$$

where we have used $|\tilde{h}_{ij}|^2 = f_\sigma H^\alpha$.

As to the two last terms [6] and [7], we evidently may estimate term [6] by

$$C\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|\,d\mu,$$

with $C = C(n, K_1, K_2)$ (see remark 5.8).

Since $H \geq H_{\min}(0)$ for all $t \in [0, T[$ we may replace H^α by $(H_{\min}(0))^\alpha$ in term [7] and estimate it from above by

$$C\int\limits_{M_t} H^2 f_\sigma^{p-1}\,d\mu.$$

Observe that this C also depends, in addition to K_1, K_2 and L, on $H_{\min}(0)$.

Altogether we obtain by collecting terms and omitting the nonnegative term [1]

$$\begin{aligned} \underbrace{\int\limits_{M_t} \frac{Z}{H^\alpha} f_\sigma^{p-1}\,d\mu}_{[1]} \ \leq\ & (\eta(p-1)+4)\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\,d\mu + \frac{p-1}{\eta}\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu \\ & + \underbrace{C\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|\,d\mu}_{[2]} + \underbrace{C\int\limits_{M_t} H^2 f_\sigma^{p-1}\,d\mu}_{[3]}, \end{aligned} \tag{104}$$

where C depends on n, K_1, K_2, and $H_{\min}(0)$.

Step 3: It remains to estimate terms [1] to [3] of the above inequality. As to term [1], we show that the relation

$$Z \geq n\varepsilon^2 H^{2+\alpha} f_\sigma = n\varepsilon^2 H^2(|A|^2 - H^2/n)$$

holds everywhere on M_t for $0 \leq t < T$.

For this purpose we choose an orthonormal basis at $p \in M_t$, such that $h_{ij} = \mathrm{diag}(\kappa_1, \ldots, \kappa_n)$. Since we have assumed that condition (35) be satisfied for $0 < \varepsilon < 1/n$, we derive that

$h_{ij} \geq \varepsilon H g_{ij}$ on $[0,T[$. Then

$$\begin{aligned} Z &= \left(\sum_{i=1}^n \kappa_i\right)\left(\sum_{j=1}^n \kappa_j^3\right) - \left(\sum_{i=1}^n \kappa_i^2\right)^2 = \sum_{i<j}^n (\kappa_i\kappa_j^3 + \kappa_j\kappa_i^3) - \sum_{i<j}^n 2\kappa_i^2\kappa_j^2 \\ &= \sum_{i<j}^n \kappa_i\kappa_j(\kappa_i-\kappa_j)^2 \geq \varepsilon^2 H^2 \sum_{i<j}^n (\kappa_i-\kappa_j)^2 = n\varepsilon^2 H^2(|A|^2 - H^2/n). \end{aligned}$$

Thus

$$\begin{aligned} n\varepsilon^2 \int\limits_{M_t} f_\sigma^p H^2\, d\mu &\leq (\eta(p-1)+4) \int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\, d\mu + \frac{p-1}{\eta}\int\limits_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\, d\mu \\ &+ C\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|\, d\mu + C\int\limits_{M_t} H^2 f_\sigma^{p-1}\, d\mu. \end{aligned}$$

Now consider terms [2] and [3] of (104). Obviously they can be absorbed by other occurring terms if we slightly adjust the powers. We do this by means of Young's inequality, which is given by

$$xy \leq \zeta x^p + \zeta^{-\frac{q}{p}} y^p, \quad \text{for } \zeta > 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

By means of this inequality we estimate

$$\begin{aligned} C\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|\, d\mu &\leq C\zeta \int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\, d\mu + C\zeta^{-1}\int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}\, d\mu \\ &\leq C\zeta \int\limits_{M_t} \frac{f_\sigma^{p-1}}{H^\alpha}|\nabla H|^2\, d\mu + C\zeta(H_{\min}(0))^{-\alpha}\left(|M_0| + \int\limits_{M_t} f_\sigma^p\, d\mu\right) \end{aligned}$$

and

$$C\int\limits_{M_t} H^2 f_\sigma^{p-1}\, d\mu \leq C\zeta \int\limits_{M_t} H^2 f_\sigma^p\, d\mu + C\zeta^{-p}\int\limits_{M_t} H^2\, d\mu.$$

Here we have used $|M_t| \leq |M_0|$. The assertion follows for appropriately small choices of ζ. □

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